CONFORMAL OPERATORS ON FORMS AND DETOUR COMPLEXES ON EINSTEIN MANIFOLDS

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ABSTRACT. For even dimensional conformal manifolds several new conformally invariant objects were found recently: invariant differential complexes related to, but distinct from, the de Rham complex (these are elliptic in the case of Riemannian signature); the cohomology spaces of these; conformally stable form spaces that we may view as spaces of conformal harmonics; operators that generalise Branson's Q-curvature; global pairings between differential form bundles that descend to cohomology pairings. Here we show that these operators, spaces, and the theory underlying them, simplify significantly on conformally Einstein manifolds. We give explicit formulae for all the operators concerned. The null spaces for these, the conformal harmonics, and the cohomology spaces are expressed explicitly in terms of direct sums of subspaces of eigenspaces of the form Laplacian. For the case of non-Ricci flat spaces this applies in all signatures and without topological restrictions. In the case of Riemannian signature and compact manifolds, this leads to new results on the global invariant pairings, including for the integral of Q-curvature against the null space of the dimensional order conformal Laplacian of Graham et al..

1. INTRODUCTION

Differential forms provide a fundamental domain for the study of smooth manifolds. In Riemannian geometry the de Rham complex, its associated Hodge theory, and distinguished forms representing characteristic classes are among the most basic and important tools (e.g. [14, 15]). In physics the study of forms is partly motivated by Maxwell theory and its generalisations. Operators on differential forms feature strongly in string and brane theories. In both mathematics and physics Einstein manifolds have a central position [2] and thus they give an important class of special structures for the study of geometric objects.

Among the differential operators that are natural for pseudo–Riemannian structures only a select class are conformally invariant. Conformal invariance is a subtle property which reflects an independence of point dependent scale. This symmetry is manifest in the equations of massless particles. It is linked to CR geometry (hence complex analysis) through the Fefferman metric [19]; the natural equations on the Fefferman space are conformally invariant. This symmetry also underpins the conformal approach to Riemannian geometry. For example, it is essentially exploited in the Yamabe problem (see [36] and references therein) of prescribing the scalar curvature. Recently there has been a focus on variations of this idea, including the conformal prescription of Branson's Q-curvature [5, 13, 16]. These problems use the conformal Laplacian on functions (or densities) and its higher order analogues due to Paneitz, Graham et al. [32].

The use of conformal operators on forms provides a setting where, on the one hand, there is potential to formally generalise such theories, but which, on the other hand, should yield access to rather different geometric data. An immediate difficulty is that forms are more difficult to work with than functions and so, while there was much early work in this direction (e.g. [3, 4]), this did not yield a clear picture. In dimension 4, and inspired by constructions from twistor theory, some rather interesting directions and applications to global geometry were pioneered in the work of Eastwood and Singer [17, 18]. Links between this result and the tractor calculus of [1, 9, 10] were established in [6]. On the other hand in [11, 27] it is shown that the conformal tractor connection may be recovered as a suitable linearisation of the ambient metric of Fefferman and Graham [20] (and see also [21]). Exploiting both developments a rather complete theory of conformal operators on forms was derived in the joint works [7, 8] of the first author with Branson. The main point of that article was not simply to construct conformal operators on differential forms, but rather, to expose and develop the discovery of preferred versions of such operators and the rather elegant picture that these yield: one may immediately construct, on even dimensional conformal manifolds, a host of new global conformal invariants. Some of these generalise, in a natural way, the integral of Q-curvature.

For most of these new operators explicit formulae are not available. For any particular operator a formula may be obtained algorithmically via tractor calculus and the theory developed in [27, 28]. However the resulting operators, when presented in the usual way, are given by extremely complicated formulae. It turns out there are striking simplifications when these operators are studied on conformally Einstein manifolds. The purpose of this article is to expose this, via a comprehensive but concise treatment, and use the results to study, in the Einstein setting, the related global conformal invariants and spaces.

To describe the content in more detail we first review the relevant results from [7] and [8]. On conformal manifolds of even dimension $n \geq 4$ there is a family of formally self-adjoint conformally invariant differential complexes:

$$\mathcal{E}^0 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}^{k-1} \xrightarrow{d} \mathcal{E}^k \xrightarrow{L_k} \mathcal{E}_k \xrightarrow{\delta} \mathcal{E}_{k-1} \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathcal{E}_0 \ . \tag{1}$$

Here, for each $k \in \{0, 1, \cdots, n/2-1\}$, $\mathcal{E}^k$ denotes the space of k-forms, $\mathcal{E}_k$ denotes an appropriate density twisting of that space, d is the exterior derivative and δ its formal adjoint. An interesting feature of these complexes is that the operators L_k have the structure of a composition

$$L_k = \delta Q_{k+1} d$$

where Q_{k+1} is from a family of differential operators, parametrised by $k = -1, \cdots, n/2-1$, and which, as operators on closed forms, generalise Branson's Q-curvature; in particular under conformal rescaling of the metric $g \mapsto \widehat{g} = e^{2\omega} g$ ($\omega \in C^\infty$) these have the conformal transformation formula

$$\hat{Q}_k u = Q_k u + L_k(\omega u) \quad \text{for } u \text{ a } \textit{closed } k\text{-form},$$

$Q_0 1$ is the Q-curvature and L_0 is the dimension order GJMS operator of [32]. On closed forms these *Q-operators* have the form $Q_{k+1} = (d\delta)^{n/2-k-1}$+*lower order terms*, so in the case of Riemannian signature the complexes (1) are elliptic. Writing H^k_L

for the (conformally invariant) cohomology at k, for the complex (1), it follows that on compact Riemannian manifolds H^k_L is finite.

The composition $G_k := \delta Q_k$ is a conformal gauge companion operator for L_k and also for the exterior derivative d. What this means is that the systems (L_k, G_k) and (d, G_k) are, in a suitable sense, conformally invariant and, in the case of Riemannian signature, are graded injectively elliptic. For example the null space $\mathcal{H}^k_G$ of (d, G_k) is conformally stable and, as pointed out in [7], is a candidate for a space of "conformal harmonics". Some perspective on these objects is given by the sequence

$$(2) \qquad 0 \to H^{k-1} \to H^{k-1}_L \xrightarrow{d} \mathcal{H}^k_G \to H^k, \qquad k \in \{1, 2, \cdots, n/2\}$$

where H^k indicates the usual de Rham cohomology. The map $H^{k-1} \to H^{k-1}_L$ is the obvious inclusion, since L factors through the exterior derivative. The map $\mathcal{H}^k_G \to H^k$ is that which simply takes solutions of (d, G_k) to their cohomology class. It is immediate from the definitions of the spaces that the sequence is exact, but it is an open question whether in general the final map $\mathcal{H}^k_G \to H^k$ is surjective. When it is we say that the space is $(k-1)$-regular [7].

We present here a study of all of these spaces and operators specialised to the setting of an Einstein structure. By exploiting some recent developments we obtain a treatment which, surprisingly, obtains most of the results in a uniform way in all signatures and without assuming the manifold is compact. As indicated above the motivation is manifold. The cohomology spaces, and related structures, mentioned above are clearly fundamental to conformal geometry. An important problem is to discover what data they capture. On the other hand there is the opportunity to shed light on Einstein structures which form an important class of geometries which remain rather mysterious; for example there are very few non-existence results for compact Riemannian Einstein spaces, while the construction of examples is primarily through Kähler geometry. The idea that this might be a rewarding approach is suggested by the intimate relationship between conformal geometry and Einstein structures. Conformal structures admit a natural conformally invariant connection on a prolonged structure: this is the Cartan connection of [12], or equivalently the induced structure is the conformal standard tractor connection that was already mentioned. An Einstein structure is equivalent to a suitably generic parallel section of this tractor bundle and so is, in this sense, a type of symmetry of conformal structure.

For case of the conformal Laplacian type operators this last point was exploited heavily in [24] where two of the three main results are as follows: on Einstein manifolds the GJMS operators of [32] factor into compositions of operators each of which is of the form of a constant potential Helmholtz Laplacian; the Q-curvature is constant and (up to a universal constant) simply a power of the scalar curvature. (See also [21] where similar results are obtained using different techniques.) Here we develop the analogous theory for operators on differential forms. In fact we do much more. A first step is that we obtain factorisations of the key operators which generalise those from [24] (to our knowledge such factorisations are new even for the conformally flat Einstein setting). On the other hand in [30, 31] we show that if the

factors $P_i : \mathcal{V} \to \mathcal{V}$ (for some vector space $\mathcal{V}$), in a composition $P := P_0 P_1 \cdots P_\ell$ of mutually commuting operators, are suitably "relatively invertible" then the general inhomogeneous problem $Pu = f$ decomposes into an equivalent system $P_i u_i = f$, $i = 0, \cdots, \ell$. This is used extensively in the current work to reduce, on non-Ricci flat Einstein manifolds, the generally high order conformal operators to equivalent lower order systems. The outcome is that in any signature (and without any assumption of compactness) on non-Ricci flat Einstein manifolds we can describe the spaces $\mathcal{N}(L_k)$ (the null space of L_k), and $\mathcal{H}^k_G$ explicitly as a direct sum of $\mathcal{H}^k_\sigma := \mathcal{N}(d) \cap \mathcal{N}(\delta)$ and the (possibly trivial) "eigenspaces",

$$\overline{\mathcal{H}}^k_{\sigma,\lambda} := \{f \in \mathcal{E}^k \mid d\delta f = \lambda f\}, \quad \widetilde{\mathcal{H}}^k_{\sigma,\lambda} := \{f \in \mathcal{E}^k \mid \delta d f = \lambda f\},$$

for various explicitly known $\lambda \in \mathbb{R}$. (Here σ denotes the Einstein scale in the conformal class, see Section 5.) See in particular Proposition 5.3 and (25), and note that for $\lambda \neq 0$ the displayed spaces give the $\mathcal{R}(d)$ (range of d) and $\mathcal{R}(\delta)$ parts of the form Laplacian "eigenspace" $\{f \in \mathcal{E}^k \mid (d\delta + \delta d)f = \lambda f\}$. We also come to a simple decomposition for H^k_L (see expression (27)) and other conformal spaces from [7].

Stronger results are available in the compact Riemannian setting and these are summarised in Theorem 6.4. Observe, in particular, that this shows that all compact Riemannian Einstein even manifolds are k-regular for $k = 0, 1 \cdots, n/2 - 1$, and that, in this setting, $\mathcal{H}^k_\sigma$ agrees with the usual space of harmonics for the form Laplacian. From the k-regularity it follows that the global conformally invariant pairings on $\mathcal{H}^k_G$, as defined in [8], descend to a conformal quadratic form on de Rham cohomology. See Theorem 6.5, and also Proposition 6.2 which shows that, in the Einstein case, the pairing is given by a power of the scalar curvature; in fact by a formula which generalises the formula from [24] for Q-curvature on Einstein manifolds. Also in Theorem 6.5 we show that the conformal pairing, via Q-operators, of closed forms against forms in $\mathcal{N}(L_k)$ descends to a closed form pairing. See the Remark following Theorem 6.5, which emphasises that this also gives a result for the usual Q-curvature. Some of the results for compact Riemannian manifolds could be obtained by using, at the outset, the complete spectral resolution of the form Laplacian. However doing this conceals the fact that, for the most part, the same results are available even when we do not have access to diagonalisations of the basic operators.

The development is as follows. Section 2.1 summarises some basic conformal geometry, tractor results, and identities to be used. In Section 3 we construct Laplacian operators on weighted tractor bundles. This is in the spirit of [24], but there is an algebraic adjustment to the basic operators. Using these Laplacian power operators, in section 4 we derive formulae for the key operators, L_k, Q_k, and so forth, in the Einstein setting. The main result is Theorem 4.2. In fact in contrast to the construction in [7] (which heavily uses the Fefferman-Graham ambient metric), these operators are developed and defined directly using invariant tractor operators. That we recover the operators from [7] is the the main subject of Section 7. In each case the operators are given in terms of compositions of

commuting operators. This enables, in Section 5, the use of the tools from [30] as recounted in Theorem 5.1.

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2. Background: Einstein metrics and conformal geometry

We first sketch here notation and background for general conformal structures and their tractor calculus following [11, 27]. The latter is then used to describe operators that we will need acting on tractor forms and some key identities are developed. Some parts of the treatment are specialised to Einstein manifolds.

2.1. **Conformal geometry and tractor calculus.** Let M be a smooth manifold of dimension $n \geq 3$. Recall that a *conformal structure* of signature (p,q) on M is a smooth ray subbundle $\mathcal{Q} \subset S^2T^*M$ whose fibre over x consists of conformally related signature-(p,q) metrics at the point x. Sections of $\mathcal{Q}$ are metrics g on M. The principal bundle $\pi : \mathcal{Q} \to M$ has structure group $\mathbb{R}_+$, and each representation $\mathbb{R}_+ \ni x \mapsto x^{-w/2} \in \mathrm{End}(\mathbb{R})$ induces a natural line bundle on $(M,[g])$ that we term the conformal density bundle $E[w]$. We shall write $\mathcal{E}[w]$ for the space of sections of this bundle. Here and throughout, sections, tensors, and functions are always smooth. When no confusion is likely to arise, we will use the same notation for a bundle and its section space.

We write $\boldsymbol{g}$ for the *conformal metric*, that is the tautological section of $S^2T^*M \otimes E[2]$ determined by the conformal structure. This will be used to identify TM with $T^*M[2]$. For many calculations we will use abstract indices in an obvious way. Given a choice of metric g from the conformal class, we write ∇ for the corresponding Levi-Civita connection. With these conventions the Laplacian Δ is given by $\Delta = \boldsymbol{g}^{ab}\nabla_a\nabla_b = \nabla^b\nabla_b$. Note $E[w]$ is trivialised by a choice of metric g from the conformal class, and we write ∇ for the connection arising from this trivialisation. It follows immediately that (the coupled) ∇_a preserves the conformal metric.

Since the Levi-Civita connection is torsion-free, the (Riemannian) curvature $R_{ab}{}^c{}_d$ is given by $[\nabla_a, \nabla_b]v^c = R_{ab}{}^c{}_d v^d$ where $[\cdot,\cdot]$ indicates the commutator bracket. The Riemannian curvature can be decomposed into the totally trace-free Weyl curvature C_{abcd} and a remaining part described by the symmetric *Schouten tensor* P_{ab}, according to $R_{abcd} = C_{abcd} + 2\boldsymbol{g}_{c[a}P_{b]d} + 2\boldsymbol{g}_{d[b}P_{a]c}$, where $[\cdots]$ indicates antisymmetrisation over the enclosed indices. We shall write $J := P^a{}_a$. The *Cotton tensor* is defined by

$$A_{abc} := 2\nabla_{[b}P_{c]a}.$$

Under a *conformal transformation* we replace a choice of metric g by the metric $\hat{g} = e^{2\omega}g$, where ω is a smooth function. Explicit formulae for the corresponding transformation of the Levi-Civita connection and its curvatures are given in e.g.

[27]. We recall that, in particular, the Weyl curvature is conformally invariant $\widehat{C}_{abcd} = C_{abcd}$.

We next define the standard tractor bundle over $(M, [g])$. It is a vector bundle of rank $n+2$ defined, for each $g \in [g]$, by $[\mathcal{E}^A]_g = \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$. If $\widehat{g} = e^{2\Upsilon} g$, we identify $(\alpha, \mu_a, \tau) \in [\mathcal{E}^A]_g$ with $(\widehat{\alpha}, \widehat{\mu}_a, \widehat{\tau}) \in [\mathcal{E}^A]_{\widehat{g}}$ by the transformation

$$\begin{pmatrix} \widehat{\alpha} \\ \widehat{\mu}_a \\ \widehat{\tau} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \Upsilon_a & \delta_a{}^b & 0 \\ -\frac{1}{2}\Upsilon_c \Upsilon^c & -\Upsilon^b & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \mu_b \\ \tau \end{pmatrix}, \tag{3}$$

where $\Upsilon_a := \nabla_a \Upsilon$. It is straightforward to verify that these identifications are consistent upon changing to a third metric from the conformal class, and so taking the quotient by this equivalence relation defines the *standard tractor bundle* $\mathcal{T}$, or $\mathcal{E}^A$ in an abstract index notation, over the conformal manifold. (Alternatively the standard tractor bundle may be constructed as a canonical quotient of a certain 2-jet bundle or as an associated bundle to the normal conformal Cartan bundle [10].) On a conformal structure of signature (p, q), the bundle $\mathcal{E}^A$ admits an invariant metric h_{AB} of signature $(p+1, q+1)$ and an invariant connection, which we shall also denote by ∇_a, preserving h_{AB}. In a conformal scale g, these are given by

$$h_{AB} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \boldsymbol{g}_{ab} & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } \nabla_a \begin{pmatrix} \alpha \\ \mu_b \\ \tau \end{pmatrix} = \begin{pmatrix} \nabla_a \alpha - \mu_a \\ \nabla_a \mu_b + \boldsymbol{g}_{ab} \tau + P_{ab} \alpha \\ \nabla_a \tau - P_{ab} \mu^b \end{pmatrix}. \tag{4}$$

It is readily verified that both of these are conformally well-defined, i.e., independent of the choice of a metric $g \in [g]$. Note that h_{AB} defines a section of $\mathcal{E}_{AB} = \mathcal{E}_A \otimes \mathcal{E}_B$, where $\mathcal{E}_A$ is the dual bundle of $\mathcal{E}^A$. Hence we may use h_{AB} and its inverse h^{AB} to raise or lower indices of $\mathcal{E}_A$, $\mathcal{E}^A$ and their tensor products.

In computations, it is often useful to introduce the 'projectors' from $\mathcal{E}^A$ to the components $\mathcal{E}[1]$, $\mathcal{E}_a[1]$ and $\mathcal{E}[-1]$ which are determined by a choice of scale. They are respectively denoted by $X_A \in \mathcal{E}_A[1]$, $Z_{Aa} \in \mathcal{E}_{Aa}[1]$ and $Y_A \in \mathcal{E}_A[-1]$, where $\mathcal{E}_{Aa}[w] = \mathcal{E}_A \otimes \mathcal{E}_a \otimes \mathcal{E}[w]$, etc. Using the metrics h_{AB} and $\boldsymbol{g}_{ab}$ to raise indices, we define X^A, Z^{Aa}, Y^A. Then we immediately see that

$$Y_A X^A = 1, \quad Z_{Ab} Z^A{}_c = \boldsymbol{g}_{bc},$$

and that all other quadratic combinations that contract the tractor index vanish. In (3) note that $\widehat{\alpha} = \alpha$ and hence X^A is conformally invariant.

Given a choice of conformal scale, the *tractor-D operator* $D_A \colon \mathcal{E}_{B \cdots E}[w] \to \mathcal{E}_{AB \cdots E}[w-1]$ is defined by

$$D_A V := (n + 2w - 2) w Y_A V + (n + 2w - 2) Z_{Aa} \nabla^a V - X_A \Box V, \tag{5}$$

where $\Box V := \Delta V + wJV$. This also turns out to be conformally invariant as can be checked directly using the formulae above (or alternatively there are conformally invariant constructions of D, see e.g. [22]).

The curvature Ω of the tractor connection is defined by

$$[\nabla_a, \nabla_b] V^C = \Omega_{ab}{}^C{}_E V^E$$

for $V^C \in \mathcal{E}^C$. Using (4) and the formulae for the Riemannian curvature yields

$$\Omega_{abCE} = Z_C{}^c Z_E{}^e C_{abce} - 2X_{[C} Z_{E]}{}^e A_{eab} \tag{6}$$

We will also need a conformally invariant curvature quantity defined as follows (cf. [22, 23])

$$W_{BC}{}^E{}_F := \frac{3}{n-2} D^A X_{[A} \Omega_{BC]}{}^E{}_F, \tag{7}$$

where $\Omega_{BC}{}^E{}_F := Z_B{}^b Z_C{}^c \Omega_{bc}{}^E{}_F$. In a choice of conformal scale, W_{ABCE} is given by

$$\begin{aligned}&(n-4)\left(Z_A{}^a Z_B{}^b Z_C{}^c Z_E{}^e C_{abce} - 2Z_A{}^a Z_B{}^b X_{[C} Z_{E]}{}^e A_{eab}\right.\\ &\left.-2X_{[A} Z_{B]}{}^b Z_C{}^c Z_E{}^e A_{bce}\right) + 4X_{[A} Z_{B]}{}^b X_{[C} Z_{E]}{}^e B_{eb},\end{aligned} \tag{8}$$

where

$$B_{ab} := \nabla^c A_{acb} + P^{dc} C_{dacb}.$$

is known as the *Bach tensor*. From the formula (8) it is clear that W_{ABCD} has Weyl tensor type symmetries.

We will work with conformally Einstein manifolds. That is, conformal structures with an Einstein metric in the conformal class. This is the same as the existence of a non-vanishing section $\sigma \in \mathcal{E}[1]$ satisfying $\left[\nabla_{(a}\nabla_{b)_0} + P_{(ab)_0}\right]\sigma = 0$, where $(\ldots)_0$ indicates the trace-free symmetric part over the enclosed indices. Equivalently (see e.g. [1, 26]) there is a standard tractor I_A that is parallel with respect to the normal tractor connection ∇ and such that $\sigma := X_A I^A$ is non-vanishing. It follows that $I_A := \frac{1}{n} D_A \sigma = Y_A \sigma + Z_A^a \nabla_a \sigma - \frac{1}{n} X_A(\Delta + J)\sigma$, for some section $\sigma \in \mathcal{E}[1]$, and so $X^A I_A = \sigma$ is non-vanishing. If we compute in the scale σ, then for example $W_{ABCD} = (n-4) Z_A^a Z_B^b Z_C^c Z_D^d C_{abcd}$.

2.2. **Tractor forms.** Following [7] we write $\mathcal{E}^k[w]$ for the space of sections of $(\Lambda^k T^*M) \otimes E[w]$ (and $\mathcal{E}^k = \mathcal{E}^k[0]$). Further we put $\mathcal{E}_k[w] := \mathcal{E}^k[w + 2k - n]$. The space of closed k-forms shall be denoted by $\mathcal{C}^k \subseteq \mathcal{E}^k$.

In order to be explicit and efficient in calculations involving bundles of possibly high rank it is necessary to employ abstract index notation as follows. In the usual abstract index conventions one would write $\mathcal{E}_{[ab\cdots c]}$ (where there are implicitly k-indices skewed over) for the space $\mathcal{E}^k$. To simplify subsequent expressions we use the following conventions. Firstly indices labelled with sequential superscripts which are at the same level (i.e. all contravariant or all covariant) will indicate a completely skew set of indices. Formally we set $a^1 \cdots a^k = [a^1 \cdots a^k]$ and so, for example, $\mathcal{E}_{a^1 \ldots a^k}$ is an alternative notation for $\mathcal{E}^k$ while $\mathcal{E}_{a^1 \ldots a^{k-1}}$ and $\mathcal{E}_{a^2 \ldots a^k}$ both denote $\mathcal{E}^{k-1}$. Next, following [29] we abbreviate this notation via multi-indices: We will use the forms indices

$$\begin{aligned}\mathbf{a}^k &:= a^1 \cdots a^k = [a^1 \cdots a^k], \quad k \geq 0,\\ \dot{\mathbf{a}}^k &:= a^2 \cdots a^k = [a^2 \cdots a^k], \quad k \geq 1.\end{aligned}$$

If $k = 1$ then $\dot{\mathbf{a}}^k$ simply means the index is absent. The corresponding notations will be used for tractor indices so e.g. the bundle of tractor k–forms $\mathcal{E}_{[A^1 \ldots A^k]}$ will be denoted by $\mathcal{E}_{A^1 \ldots A^k}$ or $\mathcal{E}_{\mathbf{A}^k}$.

The structure of $\mathcal{E}_{\mathbf{A}^k}$ is

$$\mathcal{E}_{[A^1\cdots A^k]} = \mathcal{E}_{\mathbf{A}^k} \simeq \mathcal{E}^{k-1}[k] \oplus\!\!\!\!+ \left(\mathcal{E}^k[k] \oplus \mathcal{E}^{k-2}[k-2]\right) \oplus\!\!\!\!+ \mathcal{E}^{k-1}[k-2]; \tag{9}$$

in a choice of scale the semidirect sums $\oplus\!\!\!\!+$ may be replaced by direct sums and otherwise they indicate the composition series structure arising from the tensor powers of (3).

In a choice of metric g from the conformal class, the projectors (or splitting operators) X, Y, Z for $\mathcal{E}_A$ determine corresponding projectors $\mathbb{X}, \mathbb{Y}, \mathbb{Z}, \mathbb{W}$ for $\mathcal{E}_{\mathbf{A}^{k+1}}$, $k \geq 1$ These execute the splitting of this space into four components and are given as follows.

$$\begin{array}{llll}
\mathbb{Y}^k = \mathbb{Y}_{A^0A^1\cdots A^k}{}^{a^1\cdots a^k} & = \mathbb{Y}_{A^0\mathbf{A}^k}{}^{\mathbf{a}^k} & = Y_{A^0}Z_{A^1}{}^{a^1}\cdots Z_{A^k}{}^{a^k} & \in \mathcal{E}_{\mathbf{A}^{k+1}}^{\mathbf{a}^k}[-k-1] \\
\mathbb{Z}^k = \mathbb{Z}_{A^1\cdots A^k}{}^{a^1\cdots a^k} & = \mathbb{Z}_{\mathbf{A}^k}{}^{\mathbf{a}^k} & = Z_{A^1}{}^{a^1}\cdots Z_{A^k}{}^{a^k} & \in \mathcal{E}_{\mathbf{A}^k}^{\mathbf{a}^k}[-k] \\
\mathbb{W}^k = \mathbb{W}_{A'A^0A^1\cdots A^k}{}^{a^1\cdots a^k} & = \mathbb{W}_{A'A^0\mathbf{A}^k}{}^{\mathbf{a}^k} & = X_{[A'}Y_{A^0}Z_{A^1}{}^{a^1}\cdots Z_{A^k]}{}^{a^k} & \in \mathcal{E}_{\mathbf{A}^{k+2}}^{\mathbf{a}^k}[-k] \\
\mathbb{X}^k = \mathbb{X}_{A^0A^1\cdots A^k}{}^{a^1\cdots a^k} & = \mathbb{X}_{A^0\mathbf{A}^k}{}^{\mathbf{a}^k} & = X_{A^0}Z_{A^1}{}^{a^1}\cdots Z_{A^k}{}^{a^k} & \in \mathcal{E}_{\mathbf{A}^{k+1}}^{\mathbf{a}^k}[-k+1]
\end{array}$$

where $k \geq 0$. The superscript k in $\mathbb{Y}^k$, $\mathbb{Z}^k$, $\mathbb{W}^k$ and $\mathbb{X}^k$ shows the corresponding tensor valence. (This is slightly different than in [7], where k is the relevant tractor valence.) Note that $Y = \mathbb{Y}^0$, $Z = \mathbb{Z}^1$ and $X = \mathbb{X}^0$ and $\mathbb{W}^0 = X_{[A'}Y_{A^0]}$. From (4) we immediately see $\nabla_p Y_A = Z_A^a P_{pa}$, $\nabla_p Z_A^a = -\delta_p^a Y_A - P_p^a X_A$ and $\nabla_p X_A = Z_{Ap}$. From this we obtain the formulae (cf. [29])

$$\begin{aligned}
\nabla_p \mathbb{Y}_{A^0\mathbf{A}^k}{}^{\mathbf{a}^k} &= P_{pa_0}\mathbb{Z}_{A^0\mathbf{A}^k}^{a^0\,\mathbf{a}^k} + kP_p{}^{a^1}\mathbb{W}_{A^0\mathbf{A}^k}{}^{\dot{\mathbf{a}}^k} \\
\nabla_p \mathbb{Z}_{A^0\mathbf{A}^k}^{a^0\,\mathbf{a}^k} &= -(k+1)\delta_p^{a^0}\mathbb{Y}_{A^0\mathbf{A}^k}{}^{\mathbf{a}^k} - (k+1)P_p{}^{a^0}\mathbb{X}_{A^0\mathbf{A}^k}{}^{\mathbf{a}^k} \\
\nabla_p \mathbb{W}_{A^0\mathbf{A}^k}{}^{\dot{\mathbf{a}}^k} &= -\boldsymbol{g}_{pa^1}\mathbb{Y}_{A^0\mathbf{A}^k}{}^{\mathbf{a}^k} + P_{pa^1}\mathbb{X}_{A^0\mathbf{A}^k}{}^{a^1\dot{\mathbf{a}}^k} \\
\nabla_p \mathbb{X}_{A^0\mathbf{A}^k}{}^{\mathbf{a}^k} &= \boldsymbol{g}_{pa^0}\mathbb{Z}_{A^0\mathbf{A}^k}^{a^0\,\mathbf{a}^k} - k\delta_p^{a^1}\mathbb{W}_{A^0\mathbf{A}^k}{}^{\dot{\mathbf{a}}^k},
\end{aligned} \tag{10}$$

which determine the tractor connection on form tractors in a conformal scale. Similarly, one can compute the Laplacian Δ applied to the tractors $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$ and $\mathbb{W}$. As an operator on form tractors we have the opportunity to modify Δ by adding some amount of $W\sharp\sharp$, where $\sharp$ denotes the natural tensorial action of sections in $\mathrm{End}(\mathcal{E}^A)$. Analogously, we shall use $C\sharp\sharp$ to modify the Laplacian on forms; here $\sharp$ denotes the natural tensorial action of sections in $\mathrm{End}(\mathcal{E}^a)$. It turns out (cf. [7]) that it will be convenient for us to use the operator

$$\not\Delta = \begin{cases} \Delta + \frac{1}{n-4}W\sharp\sharp & n \neq 4 \\ \Delta & n = 4. \end{cases}$$

(Note $\Delta = \nabla^a\nabla_a$.) Since the Laplacian is of the second order, it is convenient to consider e.g. $\not\Delta\mathbb{Y}_{\mathbf{A}}^{\dot{\mathbf{a}}}\tau_{\dot{\mathbf{a}}}$ where $\tau_{\dot{\mathbf{a}}} \in \mathcal{E}_{\dot{\mathbf{a}}}[w]$. It will be sufficient for our purpose to calculate this only in an Einstein scale. For example, using (10) and then that

$P_{ab} = \boldsymbol{g}_{ab}J/n$, we have

$$\begin{aligned}\nabla^p\nabla_p\mathbb{Y}_{\mathbf{A}}^{\dot{\mathbf{a}}}\tau_{\dot{\mathbf{a}}} =&\nabla^p\big[P_{pa^1}\mathbb{Z}_{\mathbf{A}}^{\mathbf{a}}+(k-1)P_p{}^{a^2}\mathbb{W}_{\mathbf{A}}^{\ddot{\mathbf{a}}}+\mathbb{Y}_{\mathbf{A}}^{\dot{\mathbf{a}}}\nabla_p\big]\tau_{\dot{\mathbf{a}}}\\ =&-\mathbb{Y}_{\mathbf{A}}^{\dot{\mathbf{a}}}\big[\big(\delta d+d\delta+(1-\frac{2(k-1)(n-k+1)}{n})J+C\sharp\sharp\big)\tau\big]_{\dot{\mathbf{a}}}\\ &+\frac{2}{nk}\mathbb{Z}_{\mathbf{A}}^{\mathbf{a}}(Jd\tau)_{\mathbf{a}}-\frac{2(k-1)}{n}\mathbb{W}_{\mathbf{A}}^{\ddot{\mathbf{a}}}(J\delta\tau)_{\ddot{\mathbf{a}}}-\frac{n-2k+2}{n^2}\mathbb{X}_{\mathbf{A}}^{\dot{\mathbf{a}}}J^2\tau_{\dot{\mathbf{a}}},\end{aligned}$$

where, as usual, $\mathbf{A}=\mathbf{A}^k$ and $\mathbf{a}=\mathbf{a}^k$. Summarising, one can compute that in an Einstein scale we obtain

$$\begin{aligned}-\not\Delta\mathbb{Y}_{\mathbf{A}}^{\dot{\mathbf{a}}}\tau_{\dot{\mathbf{a}}} =&\ \mathbb{Y}_{\mathbf{A}}^{\dot{\mathbf{a}}}\big[\big(\delta d+d\delta+(1-\frac{2(k-1)(n-k+1)}{n})J\big)\tau\big]_{\dot{\mathbf{a}}}\\ &-\frac{2}{nk}\mathbb{Z}_{\mathbf{A}}^{\mathbf{a}}(Jd\tau)_{\mathbf{a}}+\frac{2(k-1)}{n}\mathbb{W}_{\mathbf{A}}^{\ddot{\mathbf{a}}}(J\delta\tau)_{\ddot{\mathbf{a}}}+\frac{n-2k+2}{n^2}\mathbb{X}_{\mathbf{A}}^{\dot{\mathbf{a}}}J^2\tau_{\dot{\mathbf{a}}}\\ -\not\Delta\mathbb{Z}_{\mathbf{A}}^{\mathbf{a}}\mu_{\mathbf{a}} =&-2k\mathbb{Y}_{\mathbf{A}}^{\dot{\mathbf{a}}}(\delta\mu)_{\dot{\mathbf{a}}}+\mathbb{Z}_{\mathbf{A}}^{\mathbf{a}}\big[\big(\delta d+d\delta-\frac{2k(n-k-1)}{n}J\big)\mu\big]_{\mathbf{a}}\\ &-\frac{2k}{n}\mathbb{X}_{\mathbf{A}}^{\dot{\mathbf{a}}}(J\delta\mu)_{\dot{\mathbf{a}}}\\ -\not\Delta\mathbb{W}_{\mathbf{A}}^{\ddot{\mathbf{a}}}\nu_{\ddot{\mathbf{a}}} =&\ \frac{2}{k-1}\mathbb{Y}_{\mathbf{A}}^{\dot{\mathbf{a}}}(d\nu)_{\dot{\mathbf{a}}}+\mathbb{W}_{\mathbf{A}}^{\ddot{\mathbf{a}}}\big[\big(\delta d+d\delta-\frac{2(k-3)(n-k+2)}{n}J\big)\nu\big]_{\ddot{\mathbf{a}}}\\ &-\frac{2}{n(k-1)}\mathbb{X}_{\mathbf{A}}^{\dot{\mathbf{a}}}(d\nu)_{\dot{\mathbf{a}}}\\ -\not\Delta\mathbb{X}_{\mathbf{A}}^{\dot{\mathbf{a}}}\rho_{\dot{\mathbf{a}}} =&\ (n-2k+2)\mathbb{Y}_{\mathbf{A}}^{\dot{\mathbf{a}}}\rho_{\dot{\mathbf{a}}}-2(k-1)\mathbb{W}_{\mathbf{A}}^{\ddot{\mathbf{a}}}(\delta\rho)_{\ddot{\mathbf{a}}}\\ &-\frac{2}{k}\mathbb{Z}_{\mathbf{A}}^{\mathbf{a}}(d\rho)_{\mathbf{a}}+\mathbb{X}_{\mathbf{A}}^{\dot{\mathbf{a}}}\big[\big(\delta d+d\delta+(1-\frac{2(k-1)(n-k+1)}{n})J\big)\rho\big]_{\dot{\mathbf{a}}}.\end{aligned}\tag{11}$$

if either $n=4$, $k=1$ or $n\neq 4$, cf. [37, (1.50)]. Here $\tau_{\dot{\mathbf{a}}}\in\mathcal{E}_{\dot{\mathbf{a}}}[w]$, $\mu_{\mathbf{a}}\in\mathcal{E}_{\mathbf{a}}[w]$, $\nu_{\ddot{\mathbf{a}}}\in\mathcal{E}_{\ddot{\mathbf{a}}}[w]$ and $\rho_{\dot{\mathbf{a}}}\in\mathcal{E}_{\dot{\mathbf{a}}}[w]$ where $\mathbf{a}=\mathbf{a}^k$, $k\geq 1$ and w is any conformal weight.

2.3. **Useful identities.** Here we first introduce and discuss some identities that hold on a general conformal manifold.

Recall that sequentially labelled indices are assumed to be skew over, e.g. $A^1A^2=[A^1A^2]$. The operator

$$D_{A^1A^2}=-2(w\mathbb{W}_{A^1A^2}+\mathbb{X}_{A^1A^2}{}^{a}\nabla_a)\tag{12}$$

was introduced in [23]. Also recall the definition of the tractor $W_{B^1B^2C^1C^2}$ in (7). By replacing Δ by $\not\Delta$ in (5) we obtain the conformally invariant operator

$$\not{D}_A=\begin{cases}D_A-\frac{1}{n-4}X_AW\sharp\sharp & n\neq 4\\ D_A & n=4.\end{cases}$$

The case $n\neq 4$ of this operator was introduced in [7]. In contrast to D, and surprisingly, the commutator of $\not{D}$ is algebraic (cf. the commutator of D in [27]) for $n\neq 4$:

$$[\not{D}_A,\not{D}_B]=\frac{n+2w-2}{n-4}\big[(n+2w-4)W_{AB}\sharp-(D_{AB}W)\sharp\sharp\big]f,\ n\neq 4.\tag{13}$$

This can be checked by a direct computation, or alternatively by a rather simple calculation using the ambient metric and its links to tractors as in Section 7. For $n=4$ we have $[\not{D}_A, \not{D}_B] = [D_A, D_B]$, see [27] for the latter. Note one can moreover show that for $n \neq 4$, the operator $D_A - yX_A W\sharp\sharp$, $y \in \mathbb{R}$ has algebraic commutator only for the value $y = \frac{1}{n-4}$.

Proposition 2.1. *$D_{A^1A^2}$ and $W_{B^1B^2C^1C^2}$ have the following properties:*
(i) $D_{[A^1A^2}W_{B^1B^2]C^1C^2} = 0$
(ii) $D_{A^1}{}^{P}W_{PA^2B^1B^2} = -W_{A^1A^2B^1B^2}$.

Proof. We shall use the form indices $\mathbf{A} = \mathbf{A}^2$, $\mathbf{B} = \mathbf{B}^2$ and $\mathbf{C} = \mathbf{C}^2$ throughout the proof. Both identities can be verified by the direct computation. To simplify the computation note that alternatively [37] we have

$$W_{\mathbf{BC}} = \big[(n-4)\mathbb{Z}_{\mathbf{B}}^{\mathbf{b}} - 2\mathbb{X}_{\mathbf{B}}^{b^2}\nabla^{b^1}\big]\Omega_{\mathbf{bC}} \in \mathcal{E}_{\mathbf{BC}}[-2].$$

(i) Using the relations $\mathbb{W}_{[\mathbf{A}}\mathbb{X}_{\mathbf{B}]}^{b} = \mathbb{X}_{[\mathbf{A}}^{a}\mathbb{X}_{\mathbf{B}]}^{b} = \mathbb{X}_{[\mathbf{A}}^{b}\mathbb{W}_{\mathbf{B}]} = 0$ (which follow from $X_{[A}X_{B]} = 0$) we obtain

$$\begin{aligned} D_{[\mathbf{A}}W_{\mathbf{B}]\mathbf{C}} =& -2(n-4)\mathbb{W}_{[\mathbf{A}}\mathbb{Z}_{\mathbf{B}]}^{\mathbf{b}}\Omega_{\mathbf{bC}} - 2(n-4)\mathbb{X}_{[\mathbf{A}}^{a}\mathbb{Y}_{\mathbf{B}]}^{b}\Omega_{ab\mathbf{C}} \\ &+ (n-4)\mathbb{X}_{[\mathbf{A}}^{a}\mathbb{Z}_{\mathbf{B}]}^{\mathbf{b}}\nabla_a\Omega_{\mathbf{bC}} - 2\mathbb{X}_{[\mathbf{A}}^{a}\mathbb{Z}_{\mathbf{B}]}^{\mathbf{b}}\boldsymbol{g}_{ab^1}\nabla^p\Omega_{pb^2\mathbf{C}}. \end{aligned}$$

Now clearly the first two terms on the right hand side add up to 0 and the remaining ones both vanish.

(ii) Clearly $\mathbb{W}_{A^1}{}^{P}\mathbb{Z}_{PA^2}^{a^1a^2} = \mathbb{X}_{A^1}{}^{Pa^1}\mathbb{X}_{PA^2}^{a^2} = 0$. Thus

$$\begin{aligned} D_{A^1}{}^{P}W_{PA^2\mathbf{B}} = -2\Big\{&4(n-4)\mathbb{W}_{A^1}{}^{P}\mathbb{X}_{PA^2}^{a^2}\nabla^q\Omega_{qa^2\mathbf{B}} \\ &+ (n-4)\mathbb{X}_{A^1}{}^{Pp}\big[-2\mathbb{Y}_{PA^2}^{a^2}\Omega_{pa^2\mathbf{B}} + \mathbb{Z}_{PA^2}^{a^1a^2}\nabla_p\Omega_{\mathbf{aB}}\big] \\ &+ \mathbb{X}_{A^1}{}^{Pp}\big[-2\mathbb{Z}_{PA^2}^{a^1a^2}\boldsymbol{g}_{pa^1} + 2\mathbb{W}_{PA^2}\delta_p^{a^2}\big]\nabla^q\Omega_{qa^2\mathbf{B}}\Big\}. \end{aligned}$$

Now using $\mathbb{W}_{A^1}{}^{P}\mathbb{X}_{PA^2}^{a^2} = \frac{1}{4}\mathbb{X}_{\mathbf{A}}^{a^2}$, $\mathbb{X}_{A^1}{}^{Pp}Z_{PA^2}^{a^1a^2} = \frac{1}{2}\mathbb{X}_{\mathbf{A}}^{a^2}\boldsymbol{g}^{pa^1}$ and $\mathbb{X}_{A^1}{}^{Pp}\mathbb{Y}_{PA^2}^{a^2} = -\frac{1}{4}\mathbb{Z}_{\mathbf{A}}^{pa^2} - \frac{1}{4}\mathbb{W}_{\mathbf{A}}\boldsymbol{g}^{pa^2}$ the proposition follows. □

Lemma 2.2. *(i) Let $I^A \in \mathcal{E}^A$ be a parallel tractor. Then $I^P D_P{}^Q W_{QA^2B^1B^2} = 0$ and $I^P D_{P[A^0}W_{A^1A^2]B^1B^2} = 0$.*

(ii) Let $I^A, \bar{I}^A \in \mathcal{E}^A$ be a two parallel tractors. Then $I^{A^1}\bar{I}^{A^2}D_{A^1A^2}W_{B^1B^2C^1C^2} = 0$.

Proof. Recall any parallel tractor I^A satisfies $I^{A^1}W_{A^1A^2B^1B^2} = 0$ [26]. Thus the first relation of (i) follows by applying I^{A^1} to Proposition 2.1 (ii) and the second relation of (i) by applying I^{A^1} to Proposition 2.1 (i). Similarly, (ii) follows by applying $I^{A^1}\bar{I}^{A^2}$ to Proposition 2.1 (i). □

3. Einstein manifolds: conformal Laplacian operators on tractors

We assume that the structure $(M,[g])$ is conformally Einstein, and write $\sigma \in \mathcal{E}[1]$ for some Einstein scale from the conformal class. Then $I^A := \frac{1}{n}D^A\sigma$ is parallel and $X^A I_A = \sigma$ is nonvanishing.

The operator $\Box = \Delta + wJ$ acting on tractor bundles of the weight w is invariant only if $n + 2w - 2 = 0$. On the other hand the scale σ (or equivalently I^A), yields the operator

$$\not{\Box}_\sigma := I^A \not{D}_A : \mathcal{E}_{B\cdots E}[w] \longrightarrow \mathcal{E}_{B\cdots E}[w-1] \tag{14}$$

which is well defined for any w, cf. [24]. Thus we can consider the composition $(\not{\Box}_\sigma)^p$, $p \in \mathbb{N}$ and we set $(\not{\Box}_\sigma)^0 := \mathrm{id}$. These operators generally depend on the choice of the Einstein scale but one has the following modification of [24, Theorem 3.1].

Theorem 3.1. *Let $\sigma, \bar{\sigma}$ be two Einstein scales in the conformal class and consider the operators*

$$\frac{1}{\sigma^p}(\not{\Box}_\sigma)^p, \frac{1}{\bar{\sigma}^p}(\not{\Box}_{\bar{\sigma}})^p : \mathcal{E}_{B\cdots E}[w] \longrightarrow \mathcal{E}_{B\cdots E}[w-2p],$$

for $p \in \mathbb{Z}_{\geq 0}$. If $w = p - n/2$ then $\frac{1}{\sigma^p}(\not{\Box}_\sigma)^p = \frac{1}{\bar{\sigma}^p}(\not{\Box}_{\bar{\sigma}})^p$.

Proof. Assume $w = p - n/2$ and denote the Einstein metric corresponding to σ by $g := \sigma^{-2}\boldsymbol{g}$. Then $\frac{1}{\sigma^p}(\not{\Box}_\sigma)^p$ is very similar to the operator P_p^g given by [24, (19)]. The difference is, beside the sign, that we have replaced D_A in the definition of P_p^g by $\not{D}_A$ in the definition of $\frac{1}{\sigma^p}(\not{\Box}_\sigma)^p$. Thus our statement is analogous to [24, Theorem 3.1] for the operators P_p^g. We can follow the proof of the latter theorem literally; the difference between D_A and $\not{D}_A$ appears only in the commutator in the last display in the proof of [24, Theorem 3.1]. In our case, we need the commutator $[\not{D}_A, \not{D}_B]$ instead of that display and to finish the proof it remains to show that $I^A \bar{I}^B [\not{D}_A, \not{D}_B]$ vanishes on density valued tractor fields. Here $I^A = \frac{1}{n} D^A \sigma$, $\bar{I}^B = \frac{1}{n} D^B \bar{\sigma}$ are parallel tractors corresponding to the Einstein scales σ, $\bar{\sigma}$. But this follows from (13), Lemma 2.2 (ii) and the fact $I^{A^1} W_{A^1 A^2 B^1 B^2} = 0$. $\Box$

4. Einstein manifolds: Q-operators, gauge operators and detour complexes

The main aim here is to recover, in the Einstein setting, the differential complexes (1), the Q-operators and the related conformal spaces, as defined in [7]. In that source the Fefferman-Graham ambient metric is used to generate the operators which form the "building blocks" of the theory. In contrast here, in the conformally Einstein setting, all the operators are "rediscovered" directly using the tractor operators. However in Section 7 we use the ambient metric to establish that we do have exactly the specialisation of the operators and spaces from [7]; see the comment following expression (19) and Proposition 7.1.

Here we work on an Einstein manifold in an Einstein scale $\sigma \in \mathcal{E}[1]$. The first step is the conformally invariant differential splitting operator

$$\begin{aligned} &M^{\mathbf{a}}_{\mathbf{A}} : \mathcal{E}_{\mathbf{a}} \longrightarrow \mathcal{E}_{\mathbf{A}}[-k] \\ &M^{\mathbf{a}}_{\mathbf{A}} f_{\mathbf{a}} = \frac{n-2k}{k} \mathbb{Z}^{\mathbf{a}}_{\mathbf{A}} f_{\mathbf{a}} + \mathbb{X}^{\dot{\mathbf{a}}}_{\mathbf{A}} (\delta f)_{\dot{\mathbf{a}}}. \end{aligned} \tag{15}$$

where $\mathbf{A} = \mathbf{A}^k$ and $\mathbf{a} = \mathbf{a}^k$. This is a special case of the operator $\overline{M}$ from [29] (up to the multiple k).

Let $I^A := \frac{1}{n}D^A\sigma$ be the Einstein tractor for $\sigma \in \mathcal{E}[1]$. Since $I^A = Y^A\sigma - \frac{1}{n}X^A\sigma J$ in the scale σ, it follows from (14) that

$$\not{D}_\sigma = \sigma(-\not{\Delta} - 2\frac{w}{n}(n+w-1)J) : \mathcal{E}_{B\cdots E}[w] \longrightarrow \mathcal{E}_{B\cdots E}[w-1] \tag{16}$$

in the scale σ. For any differential operator $E : \mathcal{E}^k \to \mathcal{E}^k$ we have the compositions

$$P_k^p[E] := \prod_{i=1}^{p} \left(E + \frac{2i(n-2k-i+1)}{n}J\right) \tag{17}$$

for $p \geq 1$. We set $P_k^p[E] := \mathrm{id}$ for $p \leq 0$. Note that P_k^p can be considered as a polynomial in E.

Next we define the operator

$$\mathbb{L}_k^p := \frac{1}{\sigma^p}(\not{D}_\sigma)^p M_{\mathbf{A}}^{\mathbf{a}} : \mathcal{E}_{\mathbf{a}}[w] \longrightarrow \mathcal{E}_{\mathbf{A}}[w-k-2p] \tag{18}$$

for $p \geq 0$ where $(\not{D}_\sigma)^0 := \mathrm{id}$. We put $\mathbb{L}_k := \mathbb{L}_k^p$ for $p = \frac{n+2w-2k}{2}$. It follows from Theorem 3.1 that $\mathbb{L}_k$ is independent on the choice of the Einstein scale σ.

Now we are ready to state the main technical step of our construction.

Theorem 4.1. *Let $f_{\mathbf{a}} \in \mathcal{E}_{\mathbf{a}}$ where $\mathbf{a} = \mathbf{a}^k$ and $p \geq 1$. Then computing in the Einstein scale σ we obtain*

$$\begin{aligned}(\mathbb{L}_k^p f)_{\mathbf{a}} = &- p(n-2k-2p)\mathbb{Y}_{\mathbf{A}}^{\dot{\mathbf{a}}}[\delta P_k^{p-1}[d\delta]f]_{\dot{\mathbf{a}}} \\ &+ \frac{1}{k}\mathbb{Z}_{\mathbf{A}}^{\mathbf{a}}[(n-2k-2p)d\delta P_k^{p-1}[d\delta]f + (n-2k)\delta d P_{k+1}^{p-1}[\delta d]f]_{\mathbf{a}} \\ &+ \mathbb{X}_{\mathbf{A}}^{\dot{\mathbf{a}}}[\delta(d\delta + \frac{p(n-2k+2)}{n}J)P_k^{p-1}[d\delta]f]_{\dot{\mathbf{a}}}.\end{aligned}$$

Proof. It is easy to show by a direct computation (using (10) and (11)) that the theorem holds for $p = 1$. Now assume, by induction, that the theorem holds for a fixed $p \in \mathbb{N}$. To verify the theorem for $p+1$ we need to compute

$$\begin{aligned}(\mathbb{L}_k^{p+1}f)_{\mathbf{a}} &= \frac{1}{\sigma^{p+1}}\not{D}_\sigma(\not{D}_\sigma)^p M_{\mathbf{A}}^{\mathbf{a}} f_{\mathbf{a}} \\ &= \frac{1}{\sigma^{p+1}}\sigma\big(-\not{\Delta} + 2\frac{k+p}{n}(n-k-p-1)J\big)(\not{D}_\sigma)^p M_{\mathbf{A}}^{\mathbf{a}} f_{\mathbf{a}} \\ &= \big(-\not{\Delta} + 2\frac{k+p}{n}(n-k-p-1)J\big)(\mathbb{L}_k^p f)_{\mathbf{a}}\end{aligned}$$

where the multiple of J follows from (16) and from the conformal weight of $(\not{D}_\sigma)^p M_{\mathbf{A}}^{\mathbf{a}} f_{\mathbf{a}} \in \mathcal{E}_{\mathbf{A}}[-k-p]$. Since the theorem yields the formula for $(\mathbb{L}_k^p f)_{\mathbf{a}}$ we can continue in the computation using (10) and (11). The rest of the proof is to finish this computation. Let us show at least the top slot. Using (10), (11) and

the formula for $(\mathbb{L}_k^p f)_{\mathbf{a}}$ we obtain

$$\begin{aligned}\mathbb{Y}_{\mathbf{A}}^{\dot{\mathbf{a}}}\Big[&- p(n-2k-2p)\delta d\delta P_k^{p-1}[d\delta]f \\ &- p(1-\frac{2(k-1)(n-k+1)}{n})(n-2k-2p)J\delta P_k^{p-1}[d\delta]f \\ &- 2(n-2k-2p)\delta d\delta Y(p-1)f \\ &+ (n-2k+2)\delta(d\delta + \frac{p(n-2k+2)}{n}J)P_k^{p-1}[d\delta]f \\ &- 2\frac{k+p}{n}(n-k-p-1)p(n-2k-2p)J\delta P_k^{p-1}[d\delta]f\Big]_{\dot{\mathbf{a}}}.\end{aligned}$$

Summing up appropriate terms, we obtain that this is exactly

$$-(p+1)(n-2k-2(p+1))\mathbb{Y}_{\mathbf{A}}^{\dot{\mathbf{a}}}[\delta P_k^p[d\delta]f]_{\dot{\mathbf{a}}}.$$

The computation of remaining slots is similar and the theorem follows. □

Now let us assume that the dimension n is even and $k \in \{1, \dots, \frac{n}{2}\}$. Then putting $p = \frac{n-2k}{2} \geq 0$ in Theorem 4.1, we obtain

$$(\mathbb{L}_k f)_{\mathbf{a}} = \frac{2p}{k}\mathbb{Z}_{\mathbf{A}}^{\mathbf{a}}\big(\delta d P_{k+1}^{p-1}[\delta d]f\big)_{\mathbf{a}} + \mathbb{X}_{\mathbf{A}}^{\dot{\mathbf{a}}}\big(\delta P_k^p[d\delta]f\big)_{\dot{\mathbf{a}}}. \tag{19}$$

for $f_{\mathbf{a}} \in \mathcal{E}_{\mathbf{a}}$. It follows from Proposition 7.1 that $\mathbb{L}_k$ acting on $\mathcal{E}_{\mathbf{a}}$ agrees with the operator $\mathbb{L}_k$ defined in [7] up to a nonzero scalar multiple. Thus we have the following results for the operators G_k^σ, Q_k^σ and L_k from [7].

Theorem 4.2. *In an Einstein scale σ we have, up to a nonzero scalar multiple, the formula*

$$\delta \prod_{i=1}^{\frac{n-2k}{2}} \left(d\delta + \frac{2i(n-2k-i+1)}{n}J\right). \tag{20}$$

for the operator G_k^σ of [7]

$$G_k^\sigma : \mathcal{E}^k \to \mathcal{E}_{k-1}.$$

As an operator on closed k-forms, up to a nonzero scalar multiple, we have:

$$Q_k^\sigma = \prod_{i=1}^{\frac{n-2k}{2}} \left(d\delta + \frac{2i(n-2k-i+1)}{n}J\right). \tag{21}$$

Note that at the $k = n/2$ extreme these are taken to mean $G_{n/2}^\sigma = \delta$ and $Q_{n/2}^\sigma = \mathrm{id}$.

Proof. Note that setting $p = \frac{n-2k}{2}$ in Theorem 4.1, the coefficient of $\mathbb{Y}$ vanishes in the display and the left-hand-side of the display is the operator $\mathbb{L}_k$ of [7]. The coefficient of $\mathbb{X}$ is thus G_k^σ as in Theorem 4.5 of [7]. This gives the formula presented. Here we have used the fact that the factor $d\delta + \frac{p(n-2k+2)}{n}J$, with $p = \frac{n-2k}{2}$, of the coefficient of $\mathbb{X}$ appears as a composition factor in $P_k^p[d\delta]$ (the factor with $i = p$ in (17)).

Now by Theorem 2.8 of [7] and its proof we have that $G_k^\sigma = \delta Q_k^\sigma$ and Q_k^σ is formally self-adjoint. Thus the formal adjoint $G_k^{\sigma,*}$ of G_k^σ is $Q_k^\sigma d : \mathcal{E}^{k-1} \to \mathcal{E}_k$. On

the other hand since Q^σ_k is a differential operator it follows from this that $G^{\sigma,*}_k$ determines the given formula for Q^σ_k as an operator on closed forms. □

Observe that from (21) we have immediately the following useful observation.

Corollary 4.3. *On Einstein manifolds and in an Einstein scale* $Q^\sigma_k : \mathcal{C}^k \to \mathcal{C}^k$.

The operator G^σ_k acting on closed forms will be denoted by $G_k : \mathcal{C}^k \to \mathcal{E}_{k-1}$. It follows from (19) that G_k is conformally invariant (recall it is defined as projection to the $\mathbb{X}$-slot of (19)).

Corollary 4.4. *In an Einstein scale* σ *the conformally invariant detour operator* $L_k : \mathcal{E}^k \to \mathcal{E}_k$ *is given, up to a nonzero scalar multiple, by*

$$L_k = \delta \prod_{i=1}^{\frac{n-2k-2}{2}} \left(d\delta + \frac{2i(n-2k-i-1)}{n} J \right) d$$

for $k < n/2$. *Moreover we set* $L_{n/2} = 0$.

Proof. On a general manifold we have $L_k = \delta Q^\sigma_{k+1} d$ from Theorem 2.8 of [7]. Hence the statement follows from the formula Q^σ_{k+1} from (21). □

Remark 4.5. Observe that the operators L_k, and the fact that they have the form $\delta M d$, may be extracted directly from expression (19). Thus, in this conformally Einstein setting, we obtain detour complexes as in (1) from (19). Of course these were developed on general conformal manifolds in [7] but the construction in the conformally Einstein setting here is independent of [7]. Proposition 7.1 is only used here to verify that it is (the specialisation of) the same complex and also to make the connection to the Q-operators from [7].

We have constructed the operator $\mathbb{L}_k$ (thus also G^σ_k, Q^σ_k and L_k) for $k \in \{1, \ldots, \frac{n}{2}\}$ as we assumed the latter range in (19). However formulae (20) for G^σ_k, (21) for Q^σ_k and Corollary 4.4 for L_k make sense also for $k = 0$ [24]; the operators Q^σ_0 and L_0 formally agree, up to a nonzero scalar multiple, with the corresponding operators Q^g and $\Box_{n/2}$ from [24]. Further we put $\mathbb{L}_0 := L_0$. Here and below $g = \sigma^{-2}\boldsymbol{g}$ is the Einstein metric. So henceforth we shall assume $k \in \{0, \ldots, \frac{n}{2}\}$.

5. Decompositions of the conformal spaces

We work on a general (possibly noncompact) conformally Einstein even dimensional manifold $(M, [g])$ of signature (p, q). As usual we write σ to denote an Einstein scale. If not stated otherwise, we assume $k \in \{0, \ldots, \frac{n}{2}\}$ and we put $\mathcal{E}_{-1} := 0$. The space $\mathcal{H}^k_G = \mathcal{N}(G_k : \mathcal{C}^k \longrightarrow \mathcal{E}_{k-1})$, where $G_k := \delta Q^\sigma_k$, is conformally invariant [7], and we shall term it the space of *conformal harmonics*.

We shall describe $\mathcal{H}^k_G$ in more details. As mentioned in the Introduction, we will use the notation

$$\overline{\mathcal{H}}^k_{\sigma,\lambda} := \{f \in \mathcal{E}^k \mid d\delta f = \lambda f\}, \quad \widetilde{\mathcal{H}}^k_{\sigma,\lambda} := \{f \in \mathcal{E}^k \mid \delta d f = \lambda f\}$$
$$\text{and} \quad \mathcal{H}^k_\sigma := \mathcal{N}(d) \cap \mathcal{N}(\delta)$$

where $\lambda \in \mathbb{R}$. Then $\mathcal{H}^k_\sigma \subseteq \overline{\mathcal{H}}^k_{\sigma,0} + \widetilde{\mathcal{H}}^k_{\sigma,0}$. Note that if $\lambda \neq 0$ then $\overline{\mathcal{H}}^k_{\sigma,\lambda} \subset \mathcal{R}(d)$, and similarly $\widetilde{\mathcal{H}}^k_{\sigma,\lambda} \subset \mathcal{R}(\delta)$.

As well as $\mathcal{H}^k_G$, we shall also study the null spaces of the operators G_k and L_k. Our treatment relies on the following observation:

Theorem 5.1. *Let $\mathcal{V}$ be a vector space over a field $\mathbb{F}$. Suppose that E is a linear endomorphism on $\mathcal{V}$, and $P = P[E] : \mathcal{V} \to \mathcal{V}$ is a linear operator polynomial in E which factors as*

$$P[E] = (E - \lambda_1) \cdots (E - \lambda_p)$$

where the scalars $\lambda_1, \ldots, \lambda_p \in \mathbb{F}$ are mutually distinct. Then the solution space $\mathcal{V}_P$, for P, admits a canonical and unique direct sum decomposition

$$\mathcal{V}_P = \oplus_{i=1}^{\ell} \mathcal{V}_{\lambda_i} \ , \tag{22}$$

where, for each i in the sum, $\mathcal{V}_{\lambda_i}$ is the solution space for $E - \lambda_i$. The projection $\mathrm{Proj}_i : \mathcal{V}_P \to \mathcal{V}_{\lambda_i}$ is given by the formula

$$\mathrm{Proj}_i = y_i \prod_{i \neq j = 1}^{j=p} (E - \lambda_j) \quad \textit{where} \quad y_i = \prod_{i \neq j=1}^{j=p} \frac{1}{\lambda_i - \lambda_j}.$$

This is a special case of Theorem 1.1 from [30]. To use this result in our setting we need the following result.

Lemma 5.2. *The constants*

$$-\frac{2i(n - 2k - i + 1)}{n}, \quad i \in \mathbb{N}, \ k \in \{0, \cdots, n-1\},$$

are mutually distinct and negative for $i = 1, \ldots, \frac{n-2k}{2}$.

Proof. Assume that the scalar

$$2i(n - 2k - i + 1) - 2j(n - 2k - j + 1) = 2(i - j)[n - 2k - (i + j) + 1]$$

is equal to zero for some $i, j = 1, \ldots, \frac{n-2k}{2}$. This can happen only if $i = j$ or $n - 2k - (i + j) + 1 = 0$. But the latter possibility cannot happen as $i, j \leq \frac{n-2k}{2}$ means $i + j \leq n - 2k$. Thus the discussed scalars are mutually distinct. The scalars are negative since for the ranges considered $i \geq 1$, and $2i \leq n - 2k$ implies that $i + 2k \leq n - i < n + 1$. □

Definition: We define the scalars:

$$\lambda^k_i := -\frac{2i(n - 2k - i + 1)}{n} J, \quad i \in \mathbb{N}, \ k \in \{0, \cdots, n-1\}, \tag{23}$$

where, recall, J is the trace of the Schouten tensor. So on Einstein manifolds these scalars are constant and, if $J \neq 0$ then these are non-zero and mutually distinct.

Proposition 5.3. *Let (M, g) be an Einstein manifold which is not Ricci flat. We will use the scalars λ^k_i from (23) and put $p = \frac{n-2k}{2}$. The null space of the conformally invariant operator $L_k : \mathcal{E}^k \to \mathcal{E}_k$ defined in Corollary 4.4 is*

$$\mathcal{N}(L_k) = \widetilde{\mathcal{H}}^k_{\sigma,0} \oplus \bigoplus_{i=1}^{p-1} \widetilde{\mathcal{H}}^k_{\sigma,\lambda^{k+1}_i}, \quad k \in \{0, \ldots, \frac{n}{2} - 1\},$$

and $\mathcal{N}(L_{n/2}) = \mathcal{E}^{n/2} = \mathcal{E}_{n/2}$.

Proof. The case $k = n/2$ is obvious so assume $k \in \{0, \dots, \frac{n}{2} - 1\}$. Since $L_k = \delta P^{p-1}_{k+1}[d\delta]d$ according to Corollary 4.4 and $P^{p-1}_{k+1}[d\delta]d = dP^{p-1}_{k+1}[\delta d]$ we get $L_k = \delta d P^{p-1}_{k+1}[\delta d]$. Now the Proposition follows from Theorem 5.1 for $E = \delta d$ and Lemma 5.2. □

Note that $\mathcal{C}^k \subseteq \widetilde{\mathcal{H}}^k_{\sigma,0}$, $\mathcal{C}^k \cap \oplus_{i=1}^{p-1} \widetilde{\mathcal{H}}^k_{\sigma,\lambda_i^{k+1}} = \{0\}$, and $\oplus_{i=1}^{p-1} \widetilde{\mathcal{H}}^k_{\sigma,\lambda_i^{k+1}} \subseteq \mathcal{R}(\delta)$ for $k \neq \frac{n}{2}$.

Lemma 5.4. *Let (M, g) be an Einstein manifold which is not Ricci flat. In the Einstein scale σ, the null space of the operator $G^\sigma_k : \mathcal{E}^k \to \mathcal{E}_{k-1}$ given by (20) is*

$$\mathcal{N}(G^\sigma_k) = \mathcal{N}(\delta) \oplus \bigoplus_{i=1}^{p} \overline{\mathcal{H}}^k_{\sigma,\lambda_i^k}, \tag{24}$$

where the scalars λ_i^k are from (23) and $p = \frac{n-2k}{2}$.

Proof. Observe $G^\sigma_k = \delta P^p_k[d\delta] = P^p_k[\delta d]\delta$. We will work in the Einstein scale σ throughout the proof. The case $k = \frac{n}{2}$ is obvious so we assume $k < \frac{n}{2}$.

Let us start the inclusion $\supseteq$. Clearly $\mathcal{N}(\delta) \subseteq \mathcal{N}(G^\sigma_k)$. Further suppose that $f \in \overline{\mathcal{H}}^k_{\sigma,\lambda_j}$ for some $j \in 1, \dots, \frac{n-2k}{2}$. This means $d\delta f = \lambda^k_j f$ using the definition of $\overline{\mathcal{H}}^k_{\sigma,\lambda^k_j}$. The composition factors in (21), which yields the formula for $P^p_k[d\delta]$, commute and one of these factors is $d\delta - \lambda^k_j$. Hence $P^p_k[d\delta]f = 0$ which means $f \in \mathcal{N}(G^\sigma_k)$.

Now we discusses the inclusion $\mathcal{N}(G^\sigma_k) \subseteq \mathcal{N}(\delta) \oplus \bigoplus_{i=1}^p \overline{\mathcal{H}}^k_{\sigma,\lambda^k_i}$. From Lemma 5.2 it follows that the λ^k_i for $i = 1, \dots, p$ are mutually distinct. First observe $\mathcal{N}(G^\sigma_k) \subseteq \mathcal{N}(d\delta P^p_k[d\delta] : \mathcal{E}^k \to \mathcal{E}^k)$ since $G^\sigma_k = \delta P^p_k[d\delta]$. It follows from Theorem 5.1 (where we put $E = d\delta$) that $f \in \mathcal{N}(d\delta P^p_k[d\delta])$ can be uniquely written in the form $f = \bar{f} + \sum_{i=1}^p f_i$ where

$$\bar{f} = \bar{y}\Big[\prod_{j=1}^{p}(d\delta - \lambda^k_j)\Big]f \quad \text{and} \quad f_i = y_i\, d\delta\Big[\prod_{i\neq j=1}^{j=p}(d\delta - \lambda^k_j)\Big]f, \quad i = 1, \dots, \ell$$

where $\bar{y}$ and y_i are appropriate scalars, and this decomposition satisfies $d\delta f_i = \lambda^k_i f_i$ for $i = 1, \dots, p$ and $d\delta \bar{f} = 0$. Note the last display means $\bar{f} = \bar{y} P^p_k[d\delta]f$.

Now consider this decomposition for $f \in \mathcal{N}(G^\sigma_k)$. The condition $d\delta f_i = \lambda^k_i f_i$ means $f_i \in \overline{\mathcal{H}}^k_{\sigma,\lambda^k_i}$. Further applying δ to $\bar{f} = \bar{y} P^p_k[d\delta]f$, we obtain $\delta\bar{f} = \bar{y}\delta P^p_k[d\delta]f = \bar{y} G^\sigma_k f = 0$. Hence $\bar{f} \in \mathcal{N}(\delta)$. □

Theorem 5.5. *Let (M, g) be an Einstein manifold which is not Ricci flat. We use λ^k_i to denote the scalars from (23) and put $p = \frac{n-2k}{2}$. The conformally invariant space $\mathcal{H}^k_G = \mathcal{N}(G_k : \mathcal{C}^k \longrightarrow \mathcal{E}_{k-1})$ is given by the direct sum*

$$\mathcal{H}^k_G = \mathcal{H}^k_\sigma \oplus \bigoplus_{i=1}^{p} \overline{\mathcal{H}}^k_{\sigma,\lambda^k_i}. \tag{25}$$

Proof. By the definition, $\mathcal{H}^k_G$ is equal to the intersection $\mathcal{N}(G^\sigma_k) \cap \mathcal{N}(d)$. Since $\mathcal{N}(G^\sigma_k) = \mathcal{N}(\delta) \oplus \bigoplus_{i=1}^p \overline{\mathcal{H}}^k_{\sigma,\lambda^k_i}$ according to Lemma 5.4, and $\overline{\mathcal{H}}^k_{\sigma,\lambda^k_i} \subseteq \mathcal{R}(d) \subseteq \mathcal{N}(d)$, we obtain $\mathcal{H}^k_G = (\mathcal{N}(\delta) \cap \mathcal{N}(d)) \oplus \bigoplus_{i=1}^p \overline{\mathcal{H}}^k_{\sigma,\lambda^k_i}$. □

As discussed in the Introduction, a conformally invariant cohomology space may be defined:

$$H^k_L := \mathcal{N}(L_k)/\mathcal{R}(d) \ ,$$

where of course d means $d : \mathcal{E}^{k-1} \to \mathcal{E}^k$, as is clear by context. From the definitions of the various spaces it follows automatically that this fits into the complex (2) of [7]. Here and below we put $H^{-1} := 0$ and $H^{-1}_L := 0$.

Let us work on a non Ricci-flat Einstein manifold. To discuss (2) we note that we have mappings

$$\overline{\mathcal{H}}^k_{\sigma,\lambda} \xrightarrow{\delta} \widetilde{\mathcal{H}}^{k-1}_{\sigma,\lambda} \quad \text{and} \quad \widetilde{\mathcal{H}}^{k-1}_{\sigma,\lambda} \xrightarrow{d} \overline{\mathcal{H}}^k_{\sigma,\lambda}. \tag{26}$$

In the case $\lambda \neq 0$ this is a bijective correspondence as $f \overset{\delta}{\mapsto} \delta f \overset{d}{\mapsto} d\delta f = \lambda f$ for $f \in \overline{\mathcal{H}}^k_{\sigma,\lambda}$ and similarly for the opposite direction. In particular

$$d : \bigoplus_{i=1}^{\frac{n-2k}{2}} \widetilde{\mathcal{H}}^{k-1}_{\sigma,\lambda^k_i} \to \bigoplus_{i=1}^{\frac{n-2k}{2}} \overline{\mathcal{H}}^k_{\sigma,\lambda^k_i},$$

is a bijection. Using this it is easily verified that our results above are consistent with (2) in the sense that from Proposition 5.3 and (25) one verifies that (2) is exact at $\mathcal{H}^k_G$. Also since $\oplus_{i=1}^{\frac{n-2k}{2}} \widetilde{\mathcal{H}}^{k-1}_{\sigma,\lambda^k_i}$ intersects trivially with $\mathcal{R}(d)$ it follows that, in the Einstein scale σ,

$$H^{k-1}_L = \bigoplus_{i=1}^{\frac{n-2k}{2}} \widetilde{\mathcal{H}}^{k-1}_{\sigma,\lambda^k_i} \oplus \left(\widetilde{\mathcal{H}}^{k-1}_{\sigma,0}/\mathcal{R}(d)\right) \ . \tag{27}$$

The spaces H^*_L also contribute to another exact complex on even conformal manifolds,

$$0 \to H^{k-1} \to H^{k-1}_L \xrightarrow{d} \mathcal{H}^k_{\mathbb{L}} \to H^k_L \ , \qquad k \in \{0, 1, \cdots, n/2\} \tag{28}$$

which is a tautological consequence of the definition: $\mathcal{H}^k_{\mathbb{L}} = \mathcal{N}(\mathbb{L}_k)$. In analogy with (2), surjectivity of the last map is not known in general. This motivates studying $\mathcal{N}(\mathbb{L}_k)$. From the results for the null spaces of L_k and G_k we have the following.

Corollary 5.6. *The null space $\mathcal{N}(\mathbb{L}_k)$, of $\mathbb{L}_k$, is conformally invariant. On (M, g), an Einstein manifold which is not Ricci flat, $\mathcal{N}(\mathbb{L}_k)$ is given by the direct sum:*

$$\mathcal{N}(\mathbb{L}_k) = \left(\mathcal{N}(\delta) \cap \mathcal{N}(\delta d)\right) \oplus \bigoplus_{i=1}^{p-1} \widetilde{\mathcal{H}}^k_{\sigma,\lambda^{k+1}_i} \oplus \bigoplus_{i=1}^{p} \overline{\mathcal{H}}^k_{\sigma,\lambda^k_i}, \quad 0 \leq k \leq \frac{n}{2} - 1$$

where the scalars λ^k_i are given in (23) and $p = \frac{n-2k}{2}$. Further $\mathcal{N}(\mathbb{L}_{n/2}) = \mathcal{N}(\delta)$.

Proof. The case $k=0$ follows from Proposition 5.3 and the case $k=\frac{n}{2}$ is obvious. Assume $1\le k\le \frac{n}{2}-1$. The conformal invariance follows from the conformally invariance of $\mathbb{L}_k$. In the scale σ, we have $\mathcal{N}(\mathbb{L}_k)=\mathcal{N}(L_k)\cap\mathcal{N}(G^\sigma_k)$ hence we need intersection of the direct sums in the displays of Proposition 5.3 and Lemma 5.4. Since $\overline{\mathcal{H}}^k_{\sigma,\lambda^k_i}\subseteq\mathcal{N}(\delta d)=\widetilde{\mathcal{H}}^k_{\sigma,0}$ (by the definition of these spaces) for $i=1,\dots,p$, we obtain

$$\mathcal{N}(L_k)\cap\mathcal{N}(G^\sigma_k)=\mathcal{N}(\delta)\cap\Big[\widetilde{\mathcal{H}}^k_{\sigma,0}\oplus\bigoplus_{i=1}^{p-1}\widetilde{\mathcal{H}}^k_{\sigma,\lambda^{k+1}_i}\Big].$$

Since similarly $\widetilde{\mathcal{H}}^k_{\sigma,\lambda^{k+1}_i}\subseteq\mathcal{N}(\delta)$ for $i=1,\dots,p-1$, the statement follows. □

Returning to the sequence (28), exactness is easily verified using Corollary 5.6 and the bijections (26). Note the direct summand $\mathcal{N}(\delta)\cap\mathcal{N}(\delta d)$ from the previous corollary satisfies

$$\mathcal{N}(\delta)\cap\mathcal{N}(d)\subseteq\mathcal{N}(\delta)\cap\mathcal{N}(\delta d)\subseteq\mathcal{N}(d\delta+\delta d)\ . \tag{29}$$

Remark. We can also study the space $\mathcal{N}(Q^\sigma_k:\mathcal{C}^k\to\mathcal{E}_k)$. A direct application of Theorem 5.1 shows that

$$\mathcal{N}(Q^\sigma_k:\mathcal{C}^k\to\mathcal{E}_k)=\bigoplus_{i=1}^{p}\overline{\mathcal{H}}^k_{\sigma,\lambda^k_i}\subseteq\mathcal{R}(d),$$

on (M,g) an Einstein manifold which is not Ricci flat and with notation as above.

Let us, as usual, assume that (M,g) is Einstein and not Ricci flat. The operator Q^σ_k simplifies on $\mathcal{H}^k_G$. Considering (25), observe that Q^σ_k vanishes on $\overline{\mathcal{H}}^k_{\sigma,\lambda^k_i}\subseteq\mathcal{H}^k_G$, for each of the nonzero scalars λ^k_i; this is because the composition factor $d\delta-\lambda^k_i$ of Q^σ_k vanishes on $\overline{\mathcal{H}}^k_{\sigma,\lambda^k_i}$. Further, Q^σ_k is a multiple of the identity on $\mathcal{H}^k_\sigma$ because δ vanishes on $\mathcal{H}^k_\sigma$. Using (25), we summarise this.

Proposition 5.7. *Let (M,g) be an Einstein manifold which is not Ricci flat. The restriction of $Q^\sigma_k:\mathcal{C}^k\to\mathcal{E}_k$ to the conformal harmonics $\mathcal{H}^k_G$ is given in the Einstein scale σ as follows:*

$$Q^\sigma_k|_{\overline{\mathcal{H}}^k_{\sigma,\lambda^k_i}}=0\quad\textit{and}\quad Q^\sigma_k|_{\mathcal{H}^k_\sigma}=s^kJ^{(n-2k)/2}\,\mathrm{id}\quad\textit{where}$$

$$s^k=\prod_{i=1}^{\frac{n-2k}{2}}\frac{2i(n-2k-i+1)}{n}.$$

Note the last display means $s^{n/2}=1$.

The space $\mathcal{B}^k=\{df\mid Q^\sigma_k df\in\mathcal{R}(\delta)\}\subseteq\mathcal{H}^k_G$ is conformally invariant and in [7] plays a role in studying $\mathcal{H}^k_G$. Clearly $f':=df\in\overline{\mathcal{H}}^k_{\sigma,\lambda^k_i}$, $i=1,\dots,p:=\frac{n-2k}{2}$ satisfies $Q^\sigma_k f'=0$ thus trivially $f'\in\mathcal{B}^k$. Therefore $\bigoplus_{i=1}^p\overline{\mathcal{H}}^k_{\sigma,\lambda^k_i}\subseteq\mathcal{B}^k$.

6. Compact conformally-Einstein spaces

Recall that for $\varphi \in \mathcal{E}^k$, $\psi \in \mathcal{E}_k$, and $(M, [g])$ compact of signature (p, q), there is the natural conformally invariant global pairing

$$\varphi, \psi \mapsto \langle \varphi, \psi \rangle := \int_M \varphi \cdot \psi \, d\mu_{\boldsymbol{g}}, \tag{30}$$

where $\varphi \cdot \psi \in \mathcal{E}[-n]$ denotes a complete contraction between φ and ψ. When M is orientable we have

$$\langle \varphi, \psi \rangle = \int_M \varphi \wedge \star\psi$$

where $\star$ is the conformal Hodge star operator.

This pairing combines with the operator Q_k^σ to yield other global pairings. For example on compact pseudo-Riemannian manifolds, there is a conformally invariant pairing between $\mathcal{N}(L_k)$ and $\mathcal{C}^k$ given by

$$(u, w) \mapsto \langle u, Q_k w \rangle \qquad k = 0, 1, \cdots, n/2 \tag{31}$$

for $w \in \mathcal{C}^k$ and $u \in \mathcal{N}(L_k)$ [7, Theorem 2.9,(ii)]. We want to examine this in the Einstein setting. The case $k = n/2$ just recovers (30) and so we focus on the remaining cases.

We assume that (M, g) is even dimensional, Einstein and not Ricci-flat. Consider $\langle u, Q_k w \rangle$ with $w \in \mathcal{C}^k$, $u \in \mathcal{N}(L_k)$, $k \in \{0, \cdots n/2 - 1\}$. By Proposition 5.3 u decomposes directly: $u = u_0 + u_1$ where $u_0 \in \widetilde{\mathcal{H}}^k_{\sigma,0}$ and $u_1 \in \oplus_{i=1}^{p-1} \widetilde{\mathcal{H}}^k_{\sigma, \lambda_i^{k+1}}$. Now $u_1 = \delta u'$ for some $(k+1)$-form u' so, integrating by parts,

$$\langle u_1, Q_k^\sigma w \rangle = \langle u', dQ_k^\sigma w \rangle \ .$$

But using Corollary 4.3 we have $dQ_k^\sigma w = 0$. We summarise this simplification of the pairing.

Lemma 6.1. *On an even, Einstein, and non Ricci-flat, compact manifold $(M, g = \sigma^{-2}\boldsymbol{g})$ the pairing on $\mathcal{N}(L_k) \times \mathcal{C}^k$ descends to $\widetilde{\mathcal{H}}^k_{\sigma,0} \times \mathcal{C}^k$.*

Note that for $k = 0$, $\widetilde{\mathcal{H}}^k_{\sigma,0}$ is the null space of the Laplacian.

Next, on compact pseudo-Riemannian manifolds, we recall that Q_k^σ also gives a conformally invariant quadratic form on $\mathcal{H}^k_G$ [8]; this is given by (31) with now $u, w \in \mathcal{H}^k_G$. We write this as

$$\tilde{\Theta} : \mathcal{H}^k_G \times \mathcal{H}^k_G \to \mathbb{R} \ . \tag{32}$$

By Proposition 5.7, this specialises as follows:

Proposition 6.2. *On non Ricci-flat compact even Einstein manifolds (M, g) the quadratic form $\tilde{\Theta} : \mathcal{H}^k_G \times \mathcal{H}^k_G \to \mathbb{R}$ descends to*

$$\mathcal{H}^k_\sigma \times \mathcal{H}^k_\sigma \to \mathbb{R} \qquad k \in \{0, 1, \ldots, n/2\}$$

given by

$$(u, w) \mapsto s^k J^{\frac{n-2k}{2}} \langle u, w \rangle$$

where the constant s^k is given in Proposition 5.7.

6.1. **Compact Riemannian spaces.** We now assume (M,g) is a compact Einstein manifold of Riemannian signature. As above we relate g to $\sigma \in \mathcal{E}_+[1]$ by $g=\sigma^{-2}\boldsymbol{g}$, we assume $k \in \{0,\dots,\frac{n}{2}\}$ and we set $\mathcal{E}_{-1} := 0$. Many results from the previous section simplify in this setting. In particular we may use the de Rham Hodge decomposition $\mathcal{E}^k = \mathcal{R}(d) \oplus \mathcal{R}(\delta) \oplus \mathcal{H}^k_\sigma$ and $\mathcal{H}^k_\sigma$ is the usual space of de Rham harmonics, that is $\mathcal{H}^k_\sigma = \mathcal{N}(d\delta+\delta d) \cong H^k$. It also follows that the containments in (29) may be replaced by set equalities. Note also that, for example, $\mathcal{N}(\delta d) = \mathcal{N}(d)$.

Next observe that, since the operators δd and $d\delta$ are positive, we have the following from Lemma 5.2.

Proposition 6.3. *If (M,g) is a positive scalar curvature compact Riemannian Einstein manifold then*

$$\widetilde{\mathcal{H}}^{k'}_{\sigma,\lambda^k_i} = 0 \quad \textit{and} \quad \overline{\mathcal{H}}^{k'}_{\sigma,\lambda^k_i} = 0 \qquad k' \in \{0,\cdots,n\}$$

for the λ^k_i as in (23).

Using (29) and the related observations we have the following specialisations of the results of Section 5.

Theorem 6.4. *Let (M,g) be a compact Riemannian Einstein manifold of even dimension. We have the exact sequences*

$$0 \to H^{k-1} \to H^{k-1}_L \xrightarrow{d} \mathcal{H}^k_G \to H^k \to 0,$$

and

$$0 \to H^{k-1} \to H^{k-1}_L \xrightarrow{d} \mathcal{H}^k_{\mathbb{L}} \to H^k_L \to 0.$$

In particular $(M,[g])$ is $(k-1)$-regular for $k=1,\cdots,n/2$.

Assume (M,g) is not Ricci-flat. With the scalars λ^k_i as in (23) and $p=\frac{n-2k}{2}$ we have:

$$\mathcal{N}(L_k) = \mathcal{C}^k \oplus \bigoplus_{i=1}^{p-1} \widetilde{\mathcal{H}}^k_{\sigma,\lambda^{k+1}_i}, \quad k < n/2.$$

$$\mathcal{H}^k_G = \mathcal{H}^k_\sigma \oplus \bigoplus_{i=1}^{p} \overline{\mathcal{H}}^k_{\sigma,\lambda^k_i}.$$

$$\mathcal{N}(\mathbb{L}_k) = \mathcal{H}^k_\sigma \oplus \bigoplus_{i=1}^{p-1} \widetilde{\mathcal{H}}^k_{\sigma,\lambda^{k+1}_i} \oplus \bigoplus_{i=1}^{p} \overline{\mathcal{H}}^k_{\sigma,\lambda^k_i}\ , \quad k < n/2,$$

while trivially we have $\mathcal{N}(L_{n/2}) = \mathcal{E}^{n/2}$ and $\mathcal{N}(\mathbb{L}_{n/2}) = \mathcal{N}(\delta)$. In particular, in the case of positive scalar curvature: $\mathcal{N}(L_k) = \mathcal{C}^k$ and $\mathcal{N}(\mathbb{L}_k) = \mathcal{H}^k_\sigma$ for $k<n/2$, and $\mathcal{H}^k_G = \mathcal{H}^k_\sigma$.

If (M,g) is Ricci-flat then

$$\mathcal{N}(L_k) = \mathcal{C}^k \quad \textit{and} \quad \mathcal{H}^k_G = \mathcal{N}(\mathbb{L}_k) = \mathcal{H}^k_\sigma \quad \textit{for} \quad k < n/2.$$

Note that in the non Ricci-flat case $\mathcal{H}^k_G$ is formally as in (25), but now we have $\mathcal{H}^k_\sigma \cong H^k$.

The implications for the global pairings are as follows. The first statement in the Theorem follows from Lemma 6.1, Theorem 6.4, and that $\widetilde{\mathcal{H}}^k_{\sigma,0} = \mathcal{C}^k$ in the compact Riemannian setting.

Theorem 6.5. *Let (M, g) be a compact Riemannian Einstein manifold of even dimension. The pairing on $\mathcal{N}(L_k)\times\mathcal{C}^k$, by $(u, w) \mapsto \langle u, Q_k w\rangle$, with $k = 0, 1, \cdots, n/2$ descends to $\mathcal{C}^k \times \mathcal{C}^k$. By Theorem 6.4, the quadratic form $\tilde{\Theta}$ from (32) yields a conformally invariant quadratic form*

$$H^k \times H^k \to \mathbb{R}.$$

In the Einstein scale this is given by Proposition 6.2 where $\mathcal{H}^k_\sigma$ are the usual harmonics for g. In the Ricci–flat case this quadratic form is zero for $k < n/2$ and recovers (30) for $k = n/2$.

The last statement of the Theorem uses expression (21) and Theorem 6.4.

Remark 6.6. Note that, for the case of $k = 0$ and M connected, the first result of the Theorem states that for f in the null space of the dimension order GJMS operator (recall $L_0 = \Delta^{n/2} + \textit{lower order terms}$)

$$\int fQ = c\int Q.$$

where c is a unique constant such that $c - f \in \mathcal{R}(\delta^\sigma)$. (Here we write δ^σ to emphasise that, although the display is conformally invariant, to write the difference $c - f$ as a divergence requires working in the Einstein scale.)

Corollary 6.7. *Put $p := \frac{n-2k}{2}$. In an Einstein scale the space $\mathcal{B}^k$ is given as follows:*

$$\mathcal{B}^k = \begin{cases} \oplus_{j=1}^p \overline{\mathcal{H}}^k_{\sigma,\lambda^k_i} & J \neq 0 \\ 0 & J = 0. \end{cases}$$

7. The Fefferman-Graham ambient metric

Thus let us review briefly the basic relationship between the Fefferman-Graham ambient metric construction and tractor calculus as described in [11] for general conformal manifolds.

Let $\pi : \mathcal{Q} \to M$ be a conformal structure of signature (p, q). Let us use ρ to denote the $\mathbb{R}_+$ action on $\mathcal{Q}$ given by $\rho(s)(x, g_x) = (x, s^2 g_x)$. An *ambient manifold* is a smooth $(n + 2)$-manifold $\tilde{M}$ endowed with a free $\mathbb{R}_+$–action ρ and an $\mathbb{R}_+$–equivariant embedding $i : \mathcal{Q} \to \tilde{M}$. We write $\boldsymbol{X} \in \mathfrak{X}(\tilde{M})$ for the fundamental field generating the $\mathbb{R}_+$–action, that is for $f \in C^\infty(\tilde{M})$ and $u \in \tilde{M}$ we have $\boldsymbol{X} f(u) = (d/dt)f(\rho(e^t)u)|_{t=0}$.

If $i : \mathcal{Q} \to \tilde{M}$ is an ambient manifold, then an *ambient metric* is a pseudo–Riemannian metric $\boldsymbol{h}$ of signature $(p + 1, q + 1)$ on $\tilde{M}$ such that the following conditions hold:

(i) The metric $\boldsymbol{h}$ is homogeneous of degree 2 with respect to the $\mathbb{R}_+$–action, i.e. if $\mathcal{L}_{\boldsymbol{X}}$ denotes the Lie derivative by $\boldsymbol{X}$, then we have $\mathcal{L}_{\boldsymbol{X}}\boldsymbol{h} = 2\boldsymbol{h}$. (I.e. $\boldsymbol{X}$ is a homothetic vector field for h.)
(ii) For $u = (x, g_x) \in \mathcal{Q}$ and $\xi, \eta \in T_u\mathcal{Q}$, we have $\boldsymbol{h}(i_*\xi, i_*\eta) = g_x(\pi_*\xi, \pi_*\eta)$.
To simplify the notation we will usually identify $\mathcal{Q}$ with its image in $\tilde{M}$ and suppress the embedding map i.

To link the geometry of the ambient manifold to the underlying conformal structure on M one requires further conditions. In [20, 21] Fefferman and Graham treat the construction of a formal power series solution, along $\mathcal{Q}$, for the Goursat problem of finding an ambient metric $\boldsymbol{h}$ satisfying (i) and (ii) and the condition that it be Ricci flat, i.e. $\mathrm{Ric}(\boldsymbol{h}) = 0$. In even dimensions for a general conformal structure this is obstructed at finite order. However when the underlying conformal structure is (conformally) Einstein then an explicit Ricci-flat ambient metric is available [33, 34, 35]. (In fact also more generally a similar result is available for certain products of Einstein manifolds [25].) Here we shall use only the existence part of Ricci-flat ambient metric. The uniqueness of the operators we will construct is a consequence of the fact that they can be uniquely expressed in terms of the underlying conformal structure as in [11, 27].

It turns out that one may arrange that $\boldsymbol{h}$ is a metric satisfying the conditions above (i.e. (i) and (ii) and with $\boldsymbol{h}$ Ricci flat to the order possible) with $Q := \boldsymbol{h}(\boldsymbol{X}, \boldsymbol{X})$ a defining function for $\mathcal{Q}$, and $2\boldsymbol{h}(\boldsymbol{X}, \cdot) = dQ$ to all orders in both odd and even dimensions. We write $\boldsymbol{\nabla}$ for the ambient Levi-Civita connection determined by $\boldsymbol{h}$. We will use upper case abstract indices $A, B, \cdots$ for tensors on $\tilde{M}$. For example, if v^B is a vector field on $\tilde{M}$, then the ambient Riemann tensor will be denoted $\boldsymbol{R}_{AB}{}^C{}_D$ and defined by $[\boldsymbol{\nabla}_A, \boldsymbol{\nabla}_B]v^C = \boldsymbol{R}_{AB}{}^C{}_D v^D$. In this notation the ambient metric is denoted $\boldsymbol{h}_{AB}$ and, with its inverse, this is used to raise and lower indices in the usual way. We will not normally distinguish tensors related in this way even in index free notation; the meaning should be clear from the context. Thus for example we shall use $\boldsymbol{X}$ to mean both the Euler vector field $\boldsymbol{X}^A$ and the 1-form $\boldsymbol{X}_A = \boldsymbol{h}_{AB}\boldsymbol{X}^B$.

Let $\tilde{\mathcal{E}}(w)$ denote the space of functions on $\tilde{M}$ which are homogeneous of degree $w \in \mathbb{R}$ with respect to the action ρ. That is $f \in \tilde{\mathcal{E}}(w)$ means that $\boldsymbol{X}f = wf$. Similarly a tensor field F on $\tilde{M}$ is said to be *homogeneous of degree w* if $\rho(s)^*F = s^wF$ or equivalently $\mathcal{L}_{\boldsymbol{X}}F = wF$. Just as sections of $\mathcal{E}[w]$ are equivalent to functions in $\tilde{\mathcal{E}}(w)|_{\mathcal{Q}}$ we will see that the restriction of homogeneous tensor fields to $\mathcal{Q}$ have interpretations on M as weighted sections of tractor bundles [11, 27].

On the ambient tangent bundle $T\tilde{M}$ we define an action of $\mathbb{R}_+$ by $s \cdot \xi := s^{-1}\rho(s)_*\xi$. The sections of $T\tilde{M}$ which are fixed by this action are those which are homogeneous of degree -1. Let us denote by $\boldsymbol{\mathcal{T}}$ the space of such sections and write $\boldsymbol{\mathcal{T}}(w)$ for sections in $\boldsymbol{\mathcal{T}} \otimes \tilde{\mathcal{E}}(w)$, where the $\otimes$ here indicates a tensor product over $\tilde{\mathcal{E}}(0)$. Along $\mathcal{Q}$ the $\mathbb{R}_+$ action on $T\tilde{M}$ is compatible with the $\mathbb{R}_+$ action on $\mathcal{Q}$, so the quotient $(T\tilde{M}|_{\mathcal{Q}})/\mathbb{R}_+$, is a rank $n+2$ vector bundle over $\mathcal{Q}/\mathbb{R}_+ = M$; in fact this is (up to isomorphism) the normal standard tractor bundle $\mathcal{T}$ (or $\mathcal{E}^A$) [11, 27] and the composition structure of $\mathcal{T}$ reflects the vertical subbundle $T\mathcal{Q}$ in

$T\tilde{M}|_{\mathcal{Q}}$. Sections of $\mathcal{T}$ are equivalent to sections of $T\tilde{M}|_{\mathcal{Q}}$ which are homogeneous of degree -1, that is sections of $\boldsymbol{\mathcal{T}}|_{\mathcal{Q}}$. Using this relationship one sees that the ambient metric $\boldsymbol{h}$ and the ambient connection $\boldsymbol{\nabla}$ descend to, respectively the tractor metric h, and the tractor connection $\nabla^{\mathcal{T}}$. For the metric this is obvious. We discuss the connection briefly. For $U \in \mathcal{T}$, let $\tilde{U}$ be the corresponding section of $\boldsymbol{\mathcal{T}}|_{\mathcal{Q}}$. A tangent vector field ξ on M has a lift to a homogeneous degree 0 section $\tilde{\xi}$, of $T\tilde{M}|_{\mathcal{Q}}$, which is everywhere tangent to $\mathcal{Q}$. This is unique up to adding $f\boldsymbol{X}$, where $f \in \tilde{\mathcal{E}}(0)|_{\mathcal{Q}}$. We extend $\tilde{U}$ and $\tilde{\xi}$ smoothly and homogeneously to fields on $\tilde{M}$. Then we can form $\boldsymbol{\nabla}_{\tilde{\xi}}\tilde{U}$; along $\mathcal{Q}$, this is clearly independent of the extensions. Since $\boldsymbol{\nabla}_{\boldsymbol{X}}\tilde{U} = 0$, the section $\boldsymbol{\nabla}_{\tilde{\xi}}\tilde{U}$ is also independent of the choice of $\tilde{\xi}$ as a lift of ξ. Finally, the restriction of $\boldsymbol{\nabla}_{\tilde{\xi}}\tilde{U}$ is a homogeneous degree -1 section of $T\tilde{M}|_{\mathcal{Q}}$ and so determines a section of $\mathcal{T}$ which depends only on U and ξ. This is $\nabla^{\mathcal{T}}U$.

Finally we will say that an ambient tensor F is homogeneous of *weight* w if $\boldsymbol{\nabla}_{\boldsymbol{X}}F = wF$. The weight is a convenient shifting of homogeneity degree. Note, for example, that an ambient 1-form $\tilde{U}$ which is homogeneous of degree -1 is homogeneous of weight 0 and this means that $\boldsymbol{\nabla}_{\boldsymbol{X}}\tilde{U} = 0$.

7.1. **The main result.** In Section 4 several operators were defined on conformally Einstein manifolds directly using tractor calculus and the parallel tractor of the Einstein structure. On the other hand in [7] operators with the same notation were defined on general conformal manifolds via the Fefferman-Graham ambient metric, and its link to tractor calculus. The aim of this section is simply to show that these agree (up to a nonzero multiple).

Proposition 7.1. *Assume n even and $k \in \{1, \ldots, \frac{n}{2}\}$. On Einstein manifolds the operator $\mathbb{L}_k$ defined by (19) agrees with the operator with the same notation in* [7]. *The operators L_k and G_k^σ from Section 4 also agree with the operators of the same notation in* [7].

Here "agree" means the operator is the same up to a non-zero multiple, and we will not pay attention to the detail of what this constant factor is.

On the ambient manifold a special role is played by differential operators P on ambient tensor bundles which act *tangentially* along $\mathcal{Q}$, in the sense that $PQ = QP'$ for some operator P' (or equivalently $[P, Q] = QP''$ for some P''). Note that compositions of tangential operators are tangential. If tangential operators are homogeneous (i.e. the commutator with the Lie derivative $\mathcal{L}_{\boldsymbol{X}}$ recovers a constant multiple of the operator) then they descend to operators on M. An example of a tangential operator is given by

$$(n + 2w - 2)\boldsymbol{\nabla} + \boldsymbol{X}\not{\boldsymbol{\Delta}} =: \not{\boldsymbol{D}} : \boldsymbol{\mathcal{T}}^\Phi(w) \to \boldsymbol{\mathcal{T}} \otimes \boldsymbol{\mathcal{T}}^\Phi(w - 1)$$

where $\boldsymbol{\mathcal{T}}^\Phi(w)$ indicates the space of sections, homogeneous of weight w, of some ambient tensor bundle, and

$$\not{\boldsymbol{\Delta}} = \boldsymbol{\Delta} - \boldsymbol{R}\sharp\sharp.$$

Here we use the Laplacian $\boldsymbol{\Delta} := -\boldsymbol{\nabla}^A\boldsymbol{\nabla}_A$ for compatibility with [7]. We will leave the verification that $\not{\boldsymbol{D}}$ is tangential to the reader, but note this also follows from the result that $(n+2w-2)\boldsymbol{\nabla} + \boldsymbol{X}\boldsymbol{\Delta} =: \boldsymbol{D} : \boldsymbol{\mathcal{T}}^\Phi(w) \to \boldsymbol{\mathcal{T}} \otimes \boldsymbol{\mathcal{T}}^\Phi(w-1)$ is tangential

as discussed in [27, 11]. Since this is tangential and homogeneous it descends to an operator on weighted tractors. In fact it gives the usual tractor-D operator [27, 11]. The ambient $\boldsymbol{R}\sharp\sharp$ similarly descends (in dimensions $n \neq 4$) to a multiple of $W\sharp\sharp$. Thus acting on weighted tractor bundles [28]. Thus $\boldsymbol{\mathcal{T}}^\Phi(w) \to \boldsymbol{\mathcal{T}} \otimes \boldsymbol{\mathcal{T}}^\Phi(w-1)$ descends to $\mathcal{T}^\Phi(w) \to \mathcal{T} \otimes \mathcal{T}^\Phi(w-1)$ in dimensions other than 4. (Here $\mathcal{T}^\Phi$ means the tractor bundle corresponding to $\boldsymbol{\mathcal{T}}^\Phi$.) Henceforth for $(M, [g])$ of dimension 4 we take $\not{\boldsymbol{D}} := \boldsymbol{D}$, rather then the definition above.

Now if $(M, [g])$ is conformally Einstein and I a parallel tractor corresponding to an Einstein scale then along $\mathcal{Q}$ in $\tilde{M}$ we have a corresponding parallel vector field $\boldsymbol{I}$. From the explicit formula for the ambient metric over an Einstein manifold ones sees that $\boldsymbol{I}$ extends to a parallel vector field on $\tilde{M}$. (In fact when the Einstein scale is not Ricci flat then the ambient metric is given as a product of the metric cone with a line.) We have (on $\boldsymbol{\mathcal{T}}^\Phi[w]$)

$$\boldsymbol{I}^A \not{\boldsymbol{D}}_A = (n + 2w - 2)\boldsymbol{I}^A \boldsymbol{\nabla}_A + \boldsymbol{\sigma} \not{\boldsymbol{\Delta}} ,$$

where $\boldsymbol{\sigma} = \boldsymbol{I}_A \boldsymbol{X}^A \in \tilde{\mathcal{E}}(1)$. Note that $\boldsymbol{\sigma}$ is a homogeneous function on $\mathcal{Q}$ corresponding to $\sigma = I_A X^A$.

Thus if we extend a tensor field $\boldsymbol{U} \in \boldsymbol{\mathcal{T}}^\Phi(w)|_{\mathcal{Q}}$ off $\mathcal{Q}$ in such a way that $\boldsymbol{I}^A \boldsymbol{\nabla}_A \boldsymbol{U} = 0$ (which implies $\boldsymbol{U} \in \boldsymbol{\mathcal{T}}^\Phi(w)$) then we get simply

$$\boldsymbol{I}^A \not{\boldsymbol{D}}_A = \boldsymbol{\sigma} \not{\boldsymbol{\Delta}} .$$

Note that $\boldsymbol{I}^A \boldsymbol{\nabla}_A \boldsymbol{U} = 0$ can be achieved by starting with a section along $\mathcal{Q}$ and then extending off $\mathcal{Q}$ by parallel transport. The key point here is that $I^A X_A$ is non-vanishing, at least in a neighbourhood of $\mathcal{Q}$, and so $\boldsymbol{I}^A \boldsymbol{\nabla}_A$ is not tangential to $\mathcal{Q}$.

Next observe that, since $\boldsymbol{\sigma} = \boldsymbol{I}_A \boldsymbol{X}^A$ and $\boldsymbol{I}_A$ is parallel, we have

$$\boldsymbol{\nabla}_A \boldsymbol{\sigma} = \boldsymbol{I}_A \ ,$$

which is parallel. Thus

$$[\not{\boldsymbol{\Delta}}, \boldsymbol{\sigma}] = [\boldsymbol{\Delta}, \boldsymbol{\sigma}] = 2\boldsymbol{I}^A \boldsymbol{\nabla}_A \tag{33}$$

where we consider $\boldsymbol{\sigma}$ as a multiplication operator.

The following observations will be useful.

Lemma 7.2. *If $\boldsymbol{R}$ denotes the ambient curvature then $\boldsymbol{I}^A \boldsymbol{\nabla}_A \boldsymbol{R} = 0$.*

Proof. By the Bianchi identity

$$\boldsymbol{I}^A \boldsymbol{\nabla}_A \boldsymbol{R}_{BC}{}^D{}_E + \boldsymbol{I}^A \boldsymbol{\nabla}_C \boldsymbol{R}_{AB}{}^D{}_E + \boldsymbol{I}^A \boldsymbol{\nabla}_B \boldsymbol{R}_{CA}{}^D{}_E = 0.$$

But $\boldsymbol{I}$ is parallel which implies that $[\boldsymbol{\nabla}, \boldsymbol{I}] = 0$ and $\boldsymbol{I}^A \boldsymbol{R}_{AB}{}^D{}_E = 0 = \boldsymbol{I}^A \boldsymbol{R}_{CA}{}^D{}_E$. So the result follows. □

Lemma 7.3. *If $\boldsymbol{U}$ is an ambient tensor such that $\boldsymbol{I}^A \boldsymbol{\nabla}_A \boldsymbol{U} = 0$ then, for any $p \in \mathbb{N} \cup \{0\}$, $\boldsymbol{I}^A \boldsymbol{\nabla}_A (\not{\boldsymbol{\Delta}}^p \boldsymbol{U}) = 0$*

Proof. Clearly, acting on any ambient tensor, we have $[\boldsymbol{I}^A \boldsymbol{\nabla}_A, \boldsymbol{\nabla}_B] = 0$. Thus $\boldsymbol{I}^A \boldsymbol{\nabla}_A$ commutes with the Bochner Laplacian $\boldsymbol{\Delta}$. On the other hand by definition $\not{\boldsymbol{\Delta}}$ differs from the Bochner by a curvature action: $\not{\boldsymbol{\Delta}} - \boldsymbol{\Delta} = -\boldsymbol{R}\sharp\sharp$, while from the previous Lemma the ambient curvature is parallel along the flow of $\boldsymbol{I}^A \boldsymbol{\nabla}_A$. □

The main technical result we need is this.

Proposition 7.4. *For $\boldsymbol{f}$ an ambient form homogeneous of weight $k-n/2$ we have*
$$(\boldsymbol{I}^A\boldsymbol{\not{D}}_A)^k\boldsymbol{f}=\boldsymbol{\sigma}^k\boldsymbol{\not{\Delta}}^k\boldsymbol{f}\ ,$$
along $\mathcal{Q}$.

Proof. First note that both sides are tangential operators. For the right-hand-side this is proved in [7]. For the left-hand-side it holds simply because $\boldsymbol{\not{D}}$ is tangential and $\boldsymbol{I}$ is parallel on the ambient manifold. So neither side can depend on the transverse (to $\mathcal{Q}$) derivatives of the homogeneous $\boldsymbol{f}$.

Now the result is true if $k=1$. Also, calculating along $\mathcal{Q}$,
$$(\boldsymbol{I}^A\boldsymbol{\not{D}}_A)^k\boldsymbol{f}=(\boldsymbol{I}^B\boldsymbol{\not{D}}_B)^{k-1}\boldsymbol{I}^A\boldsymbol{\not{D}}_A\boldsymbol{f}$$
and so by induction
$$(\boldsymbol{I}^A\boldsymbol{\not{D}}_A)^k\boldsymbol{f}=\boldsymbol{\sigma}^{k-1}\boldsymbol{\not{\Delta}}^{k-1}\boldsymbol{I}^A\boldsymbol{\not{D}}_A\boldsymbol{f}\ .$$
Since the result is independent of transverse derivatives we may choose the extension off $\mathcal{Q}$ to suit. Thus we assume without loss of generality that $\boldsymbol{I}^A\boldsymbol{\nabla}_A\boldsymbol{f}=0$. Then $\boldsymbol{I}^A\boldsymbol{\not{D}}_A\boldsymbol{f}=\boldsymbol{\sigma}\boldsymbol{\not{\Delta}}\boldsymbol{f}$ and so
$$\boldsymbol{\sigma}^{k-1}\boldsymbol{\not{\Delta}}^{k-1}(\boldsymbol{I}^A\boldsymbol{\not{D}}_A)\boldsymbol{f}=\boldsymbol{\sigma}^{k-1}\boldsymbol{\not{\Delta}}^{k-1}(\boldsymbol{\sigma}\boldsymbol{\not{\Delta}}\boldsymbol{f}).$$
So from (33) and Lemma 7.3 the result follows. □

By Proposition 3.2 of [7], the operator $\boldsymbol{\not{\Delta}}^m$ is homogeneous and acts tangentially on ambient differential forms of weight $m-n/2$. Thus it descends to an operator that we denote $\not{\Delta}_m$ on form-tractors of weight $m-n/2$. From the above Proposition we obtain immediately the following results.

Corollary 7.5. *On conformally Einstein manifolds $(M,[g])$ the invariant operator $\not{\Delta}_m:\mathcal{T}^k[m-n/2]\to\mathcal{T}^k[-m-n/2]$, $m\in\{0,1,2,\cdots\}$, is formally self-adjoint and given by*
$$\not{\Delta}_m=\sigma^{-m}(I^A\not{D}_A)^m$$
where $\sigma^{-2}\boldsymbol{g}$ is an Einstein metric on M and $I=\frac{1}{n}D\sigma$. In odd dimensions these are natural operators. In even dimensions the same is true with the restrictions that either $m\le n/2-2$; or $m\le n/2-1$ and $k=1$; or $m\le n/2$ and $k=0$. In the conformally flat case the operators are natural with no restrictions on $m\in\{1,2,\ldots\}$.

Proof. The statements on naturality are extracted from [7]. It only remains to establish the claim that the operator is formally self-adjoint. But this is immediate from the formula for the right-hand-side from (16) because $I^A\not{D}_A=\not{\Box}_\sigma$ according to (14). □

Finally we are ready to prove the main result:

Proof of Proposition 7.1 : By expression (40) from [7] and the fact that $\not{\Delta}_m$, as in Corollary 7.5, is formally self-adjoint we have that the operator $\mathbb{L}_k$ from [7] is given by
$$\mathbb{L}_k:=\not{\Delta}_\ell\iota(\not{D})\varepsilon(X)q_k\ ,$$

where the notation is from that source. But it is a straightforward calculation to verify that, up to a non-zero multiple, $\iota(\not{D})\varepsilon(X)q_k$ is exactly the operator M from (15). (See also [37, 2.1.2 and (2.8)] where the special case $k = 2$ is treated in detail.) So the result now follows from the Corollary and (18) where $w = 0$ and $p = \frac{n-2k}{2}$. $\square$

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